MTH6140 **Linear Algebra II**

Permutations

In Chapter 3 on Determinants we use the following notions:

Definition 1 A *permutation* of $\{1,\dots,n\}$ is a bijection from the set $\{1,\dots,n\}$ to itself. The *symmetric group* S_n consists of all permutations of the set $\{1,\dots,n\}$. (There are $n!$ such permutations.)

Example 1 For $n=9$, let $\pi : \{1,2,3,4,5,6,7,8,9\} \to \{1,2,3,4,5,6,7,8,9\}$ be the permutation given by

$$\pi(1)=5\,,\quad \pi(2)=3\,,\quad \pi(3)=1\,,\quad \pi(4)=8\,,\quad \pi(5)=2\,,$$
$$\pi(6)=6\,,\quad \pi(7)=9\,,\quad \pi(8)=4\,,\quad \pi(9)=7\,.$$

The *disjoint cycle representation* of π is:

$$\pi=(1523)(48)(6)(79)\,.$$

Here (1523) expresses the fact that π maps $1\to5$, then $5\to2$, then $2\to3$, and finally $3\to1$. The subset $\{1,5,2,3\}$ is called a *cycle* of π.

Similarly, the notation (48) means that π maps 4 to 8, and then maps 8 to 4. So the subset $\{4,8\}$ is also a cycle of π.

The number 6 is mapped to itself by π, so the singleton set $\{6\}$ is itself a cycle of π, and we have the notation (6) in the above representation of π.

Finally, π maps 7 to 9, and 9 back to 7, so $\{7,9\}$ is a cycle of π, denoted by (79) in the above representation of π.

Definition 2 For any permutation $\pi\in S_n$, there is a number $\operatorname{sign}(\pi)=\pm1$, computed as follows: write π as a product of disjoint cycles, then define $\operatorname{sign}(\pi)=(-1)^{n-k}$, where k denotes the number of cycles (including cycles of length 1).

Example 2 In the example above π has $k=4$ cycles (namely (1523), (48), (6), and (79)), so

$$\operatorname{sign}(\pi)=(-1)^{9-4}=(-1)^5=-1\,.$$