MTH6140 Linear Algebra II

Assignment 5 Solutions

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1 (a) (40%) The given quadratic form can be written as the diagonal form

$$(x+s_6y+z)^2-(y-s_7z)^2+(s_8-s_9)z^2$$

(though any other diagonal form with the correct signature (see (b) for the signature of Q), together with correct working, gets full marks).

To get the above diagonal form, note that an intermediate expression for Q is

$$Q(x,y,z)=(x+s_6y+z)^2-y^2+(s_8-s_9-s_7^2)z^2+2s_7yz\,.$$

(b) (20%) From the above diagonal form we see that the coefficient of the $(x+s_6y+z)^2$ term is positive, the coefficient of the $(y-s_7z)^2$ term is negative, and the coefficient of the z^2 term is s_8-s_9. Therefore:
The signature of Q is $2-1=1$ if $s_8>s_9$.
The signature of Q is $1-1=0$ if $s_8=s_9$.
The signature of Q is $1-2=-1$ if $s_8<s_9$.

(c) (40%) There are 2 general methods for solving this, as described in lectures (see also question 3 below). Probably the quickest method is:
The transition matrix P from the vs to the ws is

$$P=\begin{bmatrix}1&0&0\\ s_3&2&0\\ 0&s_4&s_2\end{bmatrix},$$

and by Proposition 6.5 the transition matrix between the dual bases is

$$(P^{-1})^\top=\frac{1}{2s_2}\begin{bmatrix}2s_2&-s_3s_2&s_3s_4\\ 0&s_2&-s_4\\ 0&0&2\end{bmatrix}=\begin{bmatrix}1&-s_3/2&s_3s_4/(2s_2)\\ 0&1/2&-s_4/(2s_2)\\ 0&0&1/s_2\end{bmatrix}.$$

The coordinates of the gs in the basis of fs are the columns of this matrix. In other words:

$$g_1=f_1\,,$$

$$g_2 = \frac{-s_3}{2} f_1 \; + \; \frac{1}{2} f_2 \,,$$

$$g_3 = \frac{s_3 s_4}{2 s_2} f_1 \; - \; \frac{s_4}{2 s_2} f_2 \; + \; \frac{1}{s_2} f_3 \,.$$

2 (a) Q is represented by

$$\begin{bmatrix} 3 & 6 & -3 \\ 6 & 10 & -2 \\ -3 & -2 & -5 \end{bmatrix}.$$

(b) A diagonal form for Q is

$$3(x+2y-z)^2 - 2y^2 + 8yz - 8z^2 \;=\; 3(x+2y-z)^2 \;-\; 2(y-2z)^2 \,.$$

(c) Two symmetric matrices A and A' are *congruent* if there exists an invertible matrix P such that $A' = P^\top A P$.

(d) A diagonal matrix is

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

though in fact

$$\begin{bmatrix} a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

also works, for any $a, b > 0$ (as does any diagonal matrix obtained from the one above by permuting the entries a, $-b$ and 0 on the leading diagonal).

3 The first dual basis vector g_1 satisfies $g_1(w_1) = 1$, $g_1(w_2) = g_1(w_3) = 0$. If $g_1 = a_1 f_1 + a_2 f_2 + a_3 f_3$, we find

$$\begin{aligned} a_1 + a_2 + a_3 &= 1, \\ 2a_1 + a_2 + a_3 &= 0, \\ 2a_2 + a_3 &= 0, \end{aligned}$$

giving $a_1 = -1$, $a_2 = -2$, $a_3 = 4$. So

$$g_1 = -f_1 - 2f_2 + 4f_3 \,.$$

Solving two similar sets of equations gives

$$g_2 = f_1 + f_2 - 2f_3$$

and

$$g_3 = f_2 - f_3 \,.$$

Alternatively, the transition matrix P from the vs to the ws is

$$P = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix},$$

and we showed in lectures (see Proposition 6.6) that the transition matrix between the dual bases is

$$(P^{-1})^\top = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 1 & 1 \\ 4 & -2 & -1 \end{bmatrix}.$$

The coordinates of the gs in the basis of fs are the columns of this matrix.

4 (a) True.
You all have $s_2 \geq 4$, so $3s_2 \geq 12$, whereas $s_8 + 2 \leq 11 < 12 \leq 3s_2$. If we choose any basis $v_1, v_2, \ldots, v_{3s_2}$ for V, then $U = \langle v_1, \ldots, v_{s_8+2} \rangle$ (i.e. the span of the first $s_8 + 2$ vectors in the basis) is a $(s_8 + 2)$-dimensional subspace of V.

(b) False.
Every linear map has the property that $T(0) = 0$, so $0 \in \ker(T)$, so $\ker(T)$ can never be the empty set.

(c) False.
For example if v_1, v_2, v_3, v_4 are linearly independent then $\langle v_1, v_2, v_3, v_4 \rangle$ is 4-dimensional, whereas $\langle v_1 + v_2, v_3 + v_4 \rangle$ is only 2-dimensional, so these subspaces cannot be equal.

5 (a) If your $s_9 > 0$ then: the single value $\alpha = 0$.

If your $s_9 = 0$ then: all $\alpha > 0$.

To see this, note that the matrix A represents the quadratic form

$$Q(x,y) = \alpha x^2 + 2\alpha^2 xy + (\alpha^3 + s_9)y^2 = \alpha(x + \alpha y)^2 + s_9 y^2 .$$

(b) If your $s_9 > 0$ then: all $\alpha < 0$.
If your $s_9 = 0$ then: no values of α.
To see this we use the above representation of Q together with the observation (see also the lecture notes) that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is congruent to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, because $2xy = \frac{1}{2}((x+y)^2 - (x-y)^2) = u^2 - v^2$ where $u = (x+y)/\sqrt{2}$ and $v = (x-y)/\sqrt{2}$.

6 Obviously there are many possible answers to this question. Mine would have been something like:

(a) The space $\mathbb{R}^{32940534}$ equipped with the standard inner product.

(b) The map which is represented, with respect to the standard basis, by the symmetric matrix

$$\begin{bmatrix} -0.92 & 3.18 & 103 \\ 3.18 & 5.4 & -6.3 \\ 103 & -6.3 & -2 \end{bmatrix}.$$

Or, perhaps more simply, I could have chosen the map represented by a diagonal matrix such as

$$\begin{bmatrix} -0.92 & 0 & 0 \\ 0 & 5.4 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

(c) The map $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $T\left(\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} t^3 + z \\ |y| \\ \sin(t + x^2) \\ 3txyz \end{bmatrix}$ is certainly not linear.

7 (a) (i) s_3.

(ii) $s_2 s_4$.

(iii) $-s_2 s_9$.
This is because $(s_8 f_1 - s_2 f_2 + s_1 f_3 - s_4 f_4)(s_4 v_1 + s_9 v_2 + s_7 v_3 + s_8 v_4) = s_8 f_1(s_4 v_1) - s_2 f_2(s_9 v_2) + s_1 f_3(s_7 v_3) - s_4(s_8 v_4) = s_8 s_4 - s_2 s_9 + s_1 s_7 - s_4 s_8 = -s_2 s_9$ (since you all have $s_1 = 0$).

(b) First note that everyone has $s_1 = 0$ and $s_2 \neq 0$.

If you have $s_3 = s_4 = 0$ then the subspace in question is $\left\{ \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}$, and an orthonormal basis consists of the single vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (or alternatively the single vector $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$).

If at least one of s_3 and s_4 is non-zero (this is true for most of you) then the subspace in question is $\left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}$. In this case one possible orthonormal basis is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. However there are lots of other possible answers: for example $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$ is also an orthonormal basis.

8 There are several ways of solving this problem, and there is no unique solution. One can find the eigenvalues by writing down the characteristic polynomial of A, which is

$$\begin{vmatrix} x-1 & -2 & -2 \\ -2 & x-4 & -4 \\ -2 & -4 & x-4 \end{vmatrix} = x^2(x-9),$$

so the eigenvalues are 0 and 9. An eigenvector with eigenvalue 9 is found by solving the equations

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9x \\ 9y \\ 9z \end{bmatrix}.$$

We find that $[1 \quad 2 \quad 2]^\top$ is a solution. Normalising, we find that $v_1 = [1/3 \quad 2/3 \quad 2/3]^\top$ is a unit eigenvector.

Similarly the eigenvectors with eigenvalue 0 are solutions of

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which gives us the equation $x + 2y + 2z = 0$. This has two linearly independent solutions, both orthogonal to v_1. We have to choose two orthonormal vectors satisfying this equation. This can be done, for example, by choosing a basis for the space of vectors satisfying $x + 2y + 2z = 0$, say $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, and applying the Gram–Schmidt process to give $\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $\frac{1}{3\sqrt{2}} \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$.

Alternatively one can find the eigenvector $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ by inspection, and then note that (since eigenvectors of a symmetric matrix with distinct eigenvalues are orthogonal, see Corollary 8.6) the remaining eigenvectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfy $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = x + 2y + 2z = 0$, and proceed as before.