Queen Mary
University of London

MTH6140 Linear Algebra II

Assignment 5 For handing in on Monday 13th December 2010

Question 1 only *will be marked for credit. Write your name and student number at the top of your solution before handing it in. Post the solution in the* ***YELLOW*** *post-box in the* ***BASEMENT*** *of the Maths building before 15:00 on Monday.*

Let $s_1\, s_2\, s_3\, s_4\, s_5\, s_6\, s_7\, s_8\, s_9$ **denote your student number (i.e. the digits of your student number are, in order,** s_1**,** s_2**,** s_3**,** s_4**,** s_5**,** s_6**,** s_7**,** s_8**,** s_9**).**

1 (a) Suppose the real quadratic form $Q : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$Q(x,y,z) = x^2 + (s_6^2 - 1)y^2 + (1 - s_7^2 + s_8 - s_9)z^2 + 2s_6xy + 2xz + 2(s_6 + s_7)yz\,.$$

Reduce Q to diagonal form (in the sense of Theorem 6.3 in the notes), being careful to show your working.

(b) What is the signature of Q?

(c) Suppose that the vectors v_1, v_2, v_3 form a basis for $V = \mathbb{R}^3$, and that the dual basis of V^* is denoted by f_1, f_2, f_3. If a second basis w_1, w_2, w_3 for V is given by

$$w_1 = v_1 + s_3v_2, \quad w_2 = s_1v_1 + 2v_2 + s_4v_3, \quad w_3 = s_1v_1 + s_1v_2 + s_2v_3$$

then what is the basis of V^* which is dual to w_1, w_2, w_3?

2 The real quadratic form Q is defined by

$$Q(x,y,z) = 3x^2 + 12xy + 10y^2 - 6xz - 4yz - 5z^2\,.$$

(a) Which symmetric matrix A represents Q?

(b) Reduce Q to diagonal form (in the sense of Theorem 6.3 in the notes).

(c) What does it mean for two symmetric matrices to be congruent?

(d) Find a diagonal matrix which is congruent to the matrix A from part (a).

3 Suppose that the vectors v_1, v_2, v_3 form a basis for $V = \mathbb{R}^3$, and that the dual basis of V^* is denoted by f_1, f_2, f_3. If a second basis w_1, w_2, w_3 for V is given by

$$w_1 = v_1 + v_2 + v_3, \quad w_2 = 2v_1 + v_2 + v_3, \quad w_3 = 2v_2 + v_3$$

then what is the basis of V^* which is dual to w_1, w_2, w_3?

4 State whether the following are True or False, taking care to justify your answer:

(a) Every $3s_2$-dimensional vector space V contains a $(s_8 + 2)$-dimensional subspace U.

(b) There exists a linear map $T : \mathbb{R}^{s_4+1} \to \mathbb{R}^{s_5+2}$ whose kernel is the empty set.

(c) For all vectors $v_1, v_2, v_3, v_4 \in \mathbb{R}^{s_2+s_3}$, we have the equality $\langle v_1 + v_2, v_3 + v_4 \rangle = \langle v_1, v_2, v_3, v_4 \rangle$.

5 For what real values α is the symmetric real matrix $A = \begin{bmatrix} \alpha & \alpha^2 \\ \alpha^2 & \alpha^3 + s_9 \end{bmatrix}$ congruent to

(a) the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$?

(b) the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$?

6 Give an example of
(a) an inner product space, and
(b) a self-adjoint linear map on $\mathbb{R}^3$, and
(c) a map $T : \mathbb{R}^4 \to \mathbb{R}^4$ which is *not* linear,
but choosing all examples in such a way that no other student has used the same examples as their answer for this question.

7 (a) Suppose that the vectors v_1, v_2, v_3, v_4 form a basis for $V = \mathbb{R}^4$, and that the dual basis of V^* is denoted by f_1, f_2, f_3, f_4.
Evaluate the following:

(i) $f_3(s_2 v_2 + s_3 v_3)$

(ii) $(f_1 + s_2 f_4)(s_3 v_3 + s_4 v_4)$

(iii) $(s_8 f_1 - s_2 f_2 + s_1 f_3 - s_4 f_4)(s_4 v_1 + s_9 v_2 + s_7 v_3 + s_8 v_4)$

(b) Suppose that $\mathbb{R}^3$ is equipped with the standard inner product. Give an orthonormal basis for the subspace of $\mathbb{R}^3$ spanned by the vectors $\begin{bmatrix} s_1 \\ s_3 \\ 0 \end{bmatrix}, \begin{bmatrix} s_2 \\ s_4 \\ 0 \end{bmatrix}$.

8 Find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of the matrix

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}.$$