MTH6140 **Linear Algebra II**

Assignment 4 **Solutions**

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1 (a) The characteristic polynomial is $c_A(x) = \det(xI - A) = (x - s_7)(x - s_8)(x - s_9)$ and (hence) the eigenvalues are s_7, s_8, and s_9 (some of which might be equal).

(b) One method is to evaluate the three matrices $A - s_7 I$, $A - s_8 I$, and $A - s_9 I$, then multiply them together and find that their product $c_A(A)$ equals the zero matrix.

The other method is to compute the powers A^2 and A^3, then substitute them into the characteristic polynomial

$$c_A(x) = x^3 - (s_7 + s_8 + s_9)x^2 + (s_7 s_8 + s_7 s_9 + s_8 s_9)x - s_7 s_8 s_9\,,$$

and find that $A^3 - (s_7 + s_8 + s_9)A^2 + (s_7 s_8 + s_7 s_9 + s_8 s_9)A - s_7 s_8 s_9 I = 0$.

(c) Probably the best way to answer this question is to choose A' to be any matrix whose rank is the same as $\operatorname{rank}(A)$ (hence A' is equivalent to A) but whose set of eigenvalues is not the same as for A (hence A' cannot be similar to A, because similar matrices represent the same linear map (see Section 5.2 in the Notes) and their eigenvalues are precisely the eigenvalues of this map).

How to find such an A'?

Most of you will have a matrix A whose rank equals 3 (this is the case provided $\det A \neq 0$, i.e. provided all of s_7, s_8 and s_9 are non-zero). A good choice (but by no means the only choice) of A' is then

$$A' = 10 I_3 = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix},$$

(i.e. ten times the rank-3 canonical form for equivalence); note that its only eigenvalue is 10 (a triple root of $c_{A'}(x)$), so its eigenvalue set does *not* agree with that for your matrix A.

Some of you have a matrix A whose rank equals 2, in which case you could choose A' to be ten times the rank-2 canonical form for equivalence, namely

$$A' = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

since its eigenvalues are 0 and 10, which differs from the eigenvalues for your matrix A

[*Note: The choice of multiplying by 10 in the above is simply because all the eigenvalues of your A are in the range from 0 to 9. In fact there are many possible choices of A'!*]

I *think* none of you have a matrix A of rank 1.

(d) For most of you the answer is *no*, but for a few of you the answer is *yes*.

More precisely, if your $s_6 = 0$ and your s_9 equals either s_7 or s_8, then the answer is *yes*; otherwise the answer is *no*.

Justification in the case that the answer is *yes*:

Sub-case (i): If your characteristic polynomial has roots $\lambda = s_9$ and $\lambda' \neq \lambda$, then you can check that the quadratic polynomial $f(x) = (x-\lambda)(x-\lambda')$ is such that $f(A) = 0$. [This f is precisely the minimal polynomial m_A].

Sub-case (ii): If you have $s_7 = s_8 = s_9 = \lambda$ then you can check that the quadratic polynomial $f(x) = (x-\lambda)^2$ is such that $f(A) = 0$. [This f is precisely the minimal polynomial m_A].

Justification in the case that the answer is *no*: First observe that the answer is no if and only if the minimal polynomial m_A of A is *equal* to its characteristic polynomial c_A.

Sub-case (i): If your s_7, s_8 and s_9 are *distinct* then you can deduce immediately that $m_A = c_A$, since on the one hand m_A must be a factor of c_A, but on the other hand every eigenvalue of A must be a root of m_A (see Theorem 5.7).

Sub-case (ii): If you have $s_7 = s_8$, but $s_9 \neq s_8$, then you should rule out, by direct calculation, the possibility that your $m_A(x) = (x - s_8)(x - s_9)$; i.e. compute the product

$$(A - s_8 I)(A - s_9 I) = \begin{bmatrix} 0 & s_8 - s_9 & 0 \\ 0 & 0 & 0 \\ 0 & s_6 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(e) If $s_7 \neq s_8$ then A *is* diagonalisable. Otherwise, i.e. if $s_7 = s_8$, then A is *not* diagonalisable.

Justification: By Theorem 5.9 in the Notes we know that a matrix A is diagonalisable if and only if its minimal polynomial m_A is the product of *distinct* linear factors, i.e. if all its roots have multiplicity 1. The work in part (d) of this question shows that m_A does have this property, except when $s_7 = s_8$, in which case $m_A(x)$ equals either $(x - s_7)^3$ (if $s_7 = s_8 = s_9$) or $(x - s_7)^2(x - s_9)$ (if $s_9 \neq s_7$).

Alternative justification: You could compute directly the eigenvectors of A, and check whether they form a basis for $\mathbb{R}^3$ (if they do then A is diagonalisable, otherwise it is not; see Proposition 5.4).

2 (a) If v and w are any two linearly independent vectors in $\mathbb{C}^3$ then

$$U = \{sv + tw : s, t \in \mathbb{C}\}$$

is a 2-dimensional subspace of $\mathbb{C}^3$, and the vectors v, w form a basis for U (though this is of course not the only choice of basis!).

It simply remains to choose v and w in such a way that no other student will have chosen the same vectors. For example if I chose $u = \begin{bmatrix} -\pi^\pi \\ 12 \\ 1-i \end{bmatrix}$ and $v = \begin{bmatrix} -3+88i \\ 1/7 \\ 100 \end{bmatrix}$ then it is highly unlikely that anybody else would have made the same choices.

(b) Let v be any vector in V. We have to show that $P(v) = v$. Now $\mathrm{Im}(P) = V$, so there exists $v' \in V$ such that $P(v') = v$. But P is a projection, i.e. $P^2 = P$, so $P(v) = P(P(v')) = P(v') = v$, as required.

(c) You should all have found that $\left\langle \begin{bmatrix} s_8 \\ s_7 \end{bmatrix}, \begin{bmatrix} s_6 \\ s_5 \end{bmatrix}, \begin{bmatrix} s_4 \\ s_3 \end{bmatrix}, \begin{bmatrix} s_2 \\ s_1 \end{bmatrix} \right\rangle = \mathbb{C}^2$, i.e. that your vectors $\begin{bmatrix} s_8 \\ s_7 \end{bmatrix}, \begin{bmatrix} s_6 \\ s_5 \end{bmatrix}, \begin{bmatrix} s_4 \\ s_3 \end{bmatrix}, \begin{bmatrix} s_2 \\ s_1 \end{bmatrix}$ form a spanning set for $\mathbb{C}^2$. Now choosing $V = \mathbb{C}^2$ in (b) above we deduce that the *only* such P is the *identity map* on $\mathbb{C}^2$, defined by $P\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$ for all $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{C}^2$.

(d) No if $s_8 \neq s_7$, but Yes if $s_8 = s_7$.
The minimal polynomial must divide the characteristic polynomial (see Proposition 5.6), and this is clearly false if $s_8 \neq s_7$.
If $s_8 = s_7$, however, then we may choose A to be the $(s_5+2) \times (s_5+2)$ diagonal matrix whose diagonal entries are all equal to s_8.

3 Let

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_{nn} \end{bmatrix}$$

be an upper triangular matrix. We have to show that

$$\det(A) = A_{11}A_{22}\cdots A_{nn}.$$

(a) Suppose that $A_{11} = 0$. Then A has a column of zeros, so that A is not invertible, and $\det(A) = 0$; and clearly the right-hand side is also zero.

Now suppose that $A_{11} \neq 0$. Multiply the first column by A_{11}^{-1} (this column operation multiplies the determinant by A_{11}^{-1}) to get a matrix with $A_{11} = 1$. Now applying Type 1 column operations (which don't change the determinant) we can ensure that the rest of the entries in the first row are zero.

Continuing this process we find that

(i) if some $A_{ii} = 0$, then A is not invertible and $\det(A) = 0$, so that the equation holds;

(ii) if not, then we can reduce A to I by a sequence of operations which multiply the determinant by $(A_{11}A_{22}\cdots A_{nn})^{-1}$.

In case (ii), we have that

$$\det(A)(A_{11}A_{22}\cdots A_{nn})^{-1} = \det(I) = 1,$$

so $\det(A) = A_{11}A_{22}\cdots A_{nn}$, as required.

(b) Let us use the cofactor expansion along the first column. Also, we will use induction on n, assuming the result true for upper triangular matrices of size $(n-1) \times (n-1)$.

All entries in the first column of A are zero except possibly the first entry A_{11}; so there is only one term in the cofactor expansion, namely $A_{11}A^{11}$, where A^{11} is the determinant obtained by deleting the first row and column. But this is an upper triangular matrix with diagonal entries $A_{22}, \ldots, A_{nn}$. By induction, $A^{11} = A_{22}\cdots A_{nn}$, so that $\det(A) = A_{11}\cdots A_{nn}$, as required.

4 (a) True. The matrix A^3 is square, and its entries lie in K. The determinant of a square matrix with entries in K is itself an element of K.

(b) True. For example we could choose

$$A = \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(c) False when $n > 1$ (but True when $n = 1$). For example the equality fails when $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

5 The answers to each part of this question will depend on your student number. Solutions will be illustrated by using the fictitious student number 063827195.

(a) In this case

$$A = \begin{bmatrix} 8 & 6 & 0 \\ 1 & 2 & 3 \\ 5 & 9 & 7 \end{bmatrix},$$

and the cofactors $K_{ij}(A)$ are:

$K_{11}(A) = 2\cdot7-3\cdot9 = -13\,, \quad K_{12}(A) = -(1\cdot7-3\cdot5) = 8\,, \quad K_{13}(A) = 1\cdot9-2\cdot5 = -1\,,$

$K_{21}(A) = -(6\cdot7-9\cdot0) = -42\,, \quad K_{22}(A) = 8\cdot7-5\cdot0 = 56\,, \quad K_{23}(A) = -(8\cdot9-6\cdot5) = -42\,,$

$K_{31}(A) = 6\cdot3-2\cdot0 = 18\,, \quad K_{32}(A) = -(8\cdot3-1\cdot0) = -24\,, \quad K_{33}(A) = 8\cdot2-1\cdot6 = 10\,.$

The adjugate $\text{Adj}(A) = (K_{ji}(A))$ is then:

$$\text{Adj}(A) = \begin{bmatrix} -13 & -42 & 18 \\ 8 & 56 & -24 \\ -1 & -42 & 10 \end{bmatrix}$$

(b) The cofactor expansion along the first row gives

$$\det(A) = 8(2\cdot7-3\cdot9)-6(1\cdot7-3\cdot5)+0(1\cdot9-2\cdot5) = -104+48 = -56\,.$$

(c) $\det(A)$ is in this case non-zero, so A is invertible, with

$$A^{-1} = \frac{1}{\det(A)}\text{Adj}(A) = \begin{bmatrix} 13/56 & 42/56 & -18/56 \\ -8/56 & -1 & 24/56 \\ 1/56 & 42/56 & -10/56 \end{bmatrix} = \begin{bmatrix} 13/56 & 3/4 & -9/28 \\ -1/7 & -1 & 3/7 \\ 1/56 & 3/4 & -5/28 \end{bmatrix}.$$

(d) The characteristic polynomial is

$$c_A(x) = \det(xI-A) = \begin{vmatrix} x-8 & -6 & 0 \\ -1 & x-2 & -3 \\ -5 & -9 & x-7 \end{vmatrix} = (x-8)\left((x-2)(x-7)-27\right)+6(-(x-7)-15)\,,$$

which simplifies to $c_A(x) = x^3-17x^2+53x+56$.

(e)

$$A^2 = \begin{bmatrix} 70 & 60 & 18 \\ 25 & 37 & 27 \\ 84 & 111 & 76 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 710 & 702 & 306 \\ 372 & 467 & 300 \\ 1163 & 1410 & 865 \end{bmatrix},$$

so

$$c_A(A) = \begin{bmatrix} 710 & 702 & 306 \\ 372 & 467 & 300 \\ 1163 & 1410 & 865 \end{bmatrix} - 17\begin{bmatrix} 70 & 60 & 18 \\ 25 & 37 & 27 \\ 84 & 111 & 76 \end{bmatrix} + 53\begin{bmatrix} 8 & 6 & 0 \\ 1 & 2 & 3 \\ 5 & 9 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is indeed the zero matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

6 (a) V is *not* finite dimensional. The reason (which was not asked for in the question) is the following: If V were finite-dimensional, of dimension N, say, then we can find a basis B, of cardinality N, for V. Now consider the subspace U of V defined by

$$U = \{(x_1, x_2, \ldots, x_N, x_{N+1}, 0, 0, 0, 0, 0, 0, 0, \ldots) : x_1, x_2, \ldots, x_N, x_{N+1} \in \mathbb{R}\}\,.$$

It is easily seen that U is finite-dimensional, of dimension $N+1$. But B only contains N vectors, therefore cannot span the $(N+1)$-dimensional space U, hence cannot span V, a contradiction.

(b) The only 1-dimensional subspace containing this sequence is the set

$$\{(s_1t, s_2t, s_3t, s_4t, 0, 0, 0, \ldots) : t \in \mathbb{R}\}\,.$$

[For example it is $\{(0, 6t, 3t, 8t, 0, 0, 0, 0, \ldots) : t \in \mathbb{R}\}$ if your student number is 063827195].

(c) There are many 2-dimensional subspaces of V. If we choose any two linearly independent sequences, for example $v_1 = (1, 0, 0, 0, 0, \ldots)$ and $v_2 = (0, 1, 0, 0, 0, \ldots)$, then their span $\langle v_1, v_2 \rangle$ is 2-dimensional. [For this choice of v_1, v_2 we can write their span as $\langle v_1, v_2 \rangle = \{(s, t, 0, 0, 0, 0, 0, \ldots) : s, t \in \mathbb{R}\}$].

(d) U is clearly a sub*set* of V, so we only need show that U is a vector space in its own right. For this suppose that $u = (x_1, x_2, x_3, x_4, \ldots)$ and $u' = (y_1, y_2, y_3, y_4, \ldots)$ both belong to U. This means that $s_2x_2 + s_3x_3 = 0$ and $s_2y_2 + s_3y_3 = 0$. Now

$$u + u' = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4, \ldots)\,,$$

and since $s_2(x_2+y_2)+s_3(x_3+y_3) = (s_2x_2+s_3x_3)+(s_2y_2+s_3y_3) = 0+0 = 0$ we deduce that $u + u' \in U$.

Also, if $a \in \mathbb{R}$ then $au = (ax_1, ax_2, ax_3, ax_4, \ldots)$, and $s_2(ax_2) + s_3(ax_3) = a(s_2x_2 + s_3x_3) = a \cdot 0 = 0$, so $au \in U$.

So U is a vector space in its own right, hence a subspace of U.

(e) This is a subspace of V if and only if your final student digit s_9 is equal to 0.

If $s_9 = 0$ then the proof that W is a subspace is very similar to the proof of (d) above. If $s_9 \neq 0$ then note (for example) that the zero vector $(0, 0, 0, 0, 0, 0, 0, 0, \ldots)$ does not belong to W, hence W cannot be a subspace.

7 (a) The matrix is

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ s_2 & 0 & s_2 & 0 \\ 0 & s_3 & 0 & s_3 \\ s_4 & 0 & s_4 & 0 \end{bmatrix}.$$

A vector $\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$ belongs to the kernel of S if and only if $S\left(\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, which holds if and only if $x+z=0$ and $t+y=0$ (note that here we are using the fact that you all have $s_2 \neq 0$). So the two vectors $\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ are a basis for $\ker(S)$.

(b) The matrix is

$$B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Its characteristic equation is $c_B(x) = \det(xI - B) = x^2(x^2-4)$, so the eigenvalues of B, and hence of T, are 0, 2, and -2.

(c) Reasoning as in (a) shows that the eigenvalue 0 has a 2-dimensional eigenspace, spanned by $\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$.

The eigenvector for eigenvalue 2 is obtained by solving the linear equations

$$\begin{aligned} x+z &= 2t, \\ t+y &= 2x, \\ x+z &= 2y, \\ t+y &= 2z, \end{aligned}$$

which has the solution $\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, unique up to scalar multiples.

The eigenvector for eigenvalue -2 is obtained by solving the linear equations

$$\begin{aligned} x+z &= -2t, \\ t+y &= -2x, \\ x+z &= -2y, \\ t+y &= -2z, \end{aligned}$$

which has the solution $\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, unique up to scalar multiples.

So

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

is a basis for $\mathbb{R}^4$ consisting of eigenvectors of T.

Consequently $BP = PD$, and hence $P^{-1}BP = D$, where

$$P = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & -1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

8 (a) The characteristic polynomial of A is

$$c_A(x) = \begin{vmatrix} x+3 & -8 & -2 \\ 2 & x-5 & -1 \\ 2 & -2 & x-4 \end{vmatrix} = (x-1)(x-2)(x-3).$$

(b) $m_A(x) = c_A(x) = (x-1)(x-2)(x-3)$, since we know that the (monic) polynomial $m_A(x)$ must divide $c_A(x)$, and the degree of $m_A(x)$ is not greater than the degree of $c_A(x)$.

(c) The eigenvalues of A are 1, 2 and 3, since these are the roots of the characteristic equation.
Now the eigenvectors are obtained by solving linear equations. For eigenvalue 1, the equations are

$$\begin{aligned} -3x_1 + 8x_2 + 2x_3 &= x_1, \\ -2x_1 + 5x_2 + x_3 &= x_2, \\ -2x_1 + 2x_2 + 4x_3 &= x_3, \end{aligned}$$

which has the solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$, unique up to scalar multiples.

Similarly, for the eigenvalue 2, we have $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, and for the eigenvalue 3, we have $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

(d) Using (c) we have $AQ = QD$ (and hence $Q^{-1}AQ = D$), where

$$Q = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(e) The simplest way to find the projections is as follows: Let E_i be the matrix with 1 in the ith diagonal position and 0 everywhere else, and $P_i = QE_iQ^{-1}$. Then

$$\begin{aligned} P_i^2 &= (QE_iQ^{-1})(QE_iQ^{-1}) = QE_i^2Q^{-1} = QE_iQ^{-1} = P_i, \\ P_1 + P_2 + P_3 &= Q(E_1 + E_2 + E_3)Q^{-1} = QIQ^{-1} = I, \\ P_1 + 2P_2 + 3P_3 &= Q(E_1 + 2E_2 + 3E_3)Q^{-1} = QDQ^{-1} = A, \end{aligned}$$

so P_1, P_2, P_3 are the required projections. Calculation gives

$$Q^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 6 & -1 \\ 0 & -1 & 1 \end{bmatrix},$$

so

$$P_1 = \begin{bmatrix} 5 & -10 & 0 \\ 2 & -4 & 0 \\ 2 & -4 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} -4 & 12 & -2 \\ -2 & 6 & 1 \\ -2 & 6 & 1 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix}.$$

Alternatively you can find them like this. The matrix P_1 has column space spanned by $[5 \quad 2 \quad 2]^\top$, so we have

$$P_1 = \begin{bmatrix} 5x & 5y & 5z \\ 2x & 2y & 2z \\ 2x & 2y & 2z \end{bmatrix}$$

for some (as yet undetermined) x, y, z. Now use the fact that, if v_1, v_2, v_3 denote the three eigenvectors of A, then $P_1v_1 = v_1$, $P_2v_1 = 0$, $P_3v_1 = 0$, to get three equations for x, y, z and hence find P_1. The other two are found similarly.

9 (a) You all have $s_1 = 0$, so $A = \begin{bmatrix} 0 & -4 & -2s_2 \\ 1 & 4 & s_2 \\ 0 & 0 & s_3 \end{bmatrix}$, with characteristic polynomial

$$c_A(x) = \begin{vmatrix} x & 4 & 2s_2 \\ -1 & x-4 & -s_2 \\ 0 & 0 & x-s_3 \end{vmatrix} = (x - s_3)(x-2)^2.$$

For the minimal polynomial we consider separately the two cases $s_3 = 2$ and $s_3 \neq 2$:

If $s_3 = 2$ then $m_A(x) = (x-2)^2$, because $(A - 2I)^2 = 0$ (and $A - 2I \neq 0$).

If $s_3 \neq 2$ then $m_A(x) = c_A(x) = (x - s_3)(x-2)^2$, since $(A - s_3 I)(A - 2I) \neq 0$.

(b) Its rank equals 1. One way of arguing this is to note that every column is the same (and not full of zeros) so the column rank equals 1.

(c) Minimal polynomial equals $(x-n)x$ if $n > 1$, and equals $x - 1$ if $n = 1$.
The case $n = 1$ is obvious. To prove it for $n > 1$, note that both 0 and n are eigenvalues, so $(x-n)x$ must be a factor of the minimal polynomial. But A^2 is the matrix with every entry equal to n, so is equal to nA; therefore $(A - nI)A = 0$.

(d) Diagonalisable for *all* values of n, because the minimal polynomial is a product of distinct linear factors (see Theorem 5.9 in the notes).