Queen Mary
University of London

MTH6140 Linear Algebra II

Assignment 4 For handing in on Monday 29th November 2010

Question 1 only *will be marked for credit. Write your name and student number at the top of your solution before handing it in. Post the solution in the* ***YELLOW*** *post-box in the* ***BASEMENT*** *of the Maths building before 15:00 on Monday.*

Let $s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9$ **denote your student number (i.e. the digits of your student number are, in order,** $s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9$**).**

1 Let A be the real matrix

$$A = \begin{bmatrix} s_7 & 1 & 0 \\ 0 & s_8 & s_1 \\ s_6 & 0 & s_9 \end{bmatrix}.$$

(a) Compute the characteristic polynomial $c_A(x)$, and determine the eigenvalues of A.

(b) Prove *directly* (i.e. without quoting the Cayley-Hamilton Theorem) that A satisfies its characteristic equation.

(c) Write down a matrix A' which is equivalent to A but not similar to A. Briefly justify your answer.

(d) Does there exist a quadratic polynomial f such that $f(A) = 0$? Justify your answer.

(e) Is A diagonalisable (i.e. similar to a diagonal matrix)? Justify your answer.

[*Hint: For parts (d) and (e) you may wish to consult the discussion of 'minimal polynomials' in Section 5.5 of the Notes*]

2 (a) Give an example of a 2-dimensional subspace of $\mathbb{C}^3$ which is *different* from the example given by all other students answering this question, and write down a basis for your subspace.

(b) Show that if $P : V \to V$ is a projection on a vector space V, and $\mathrm{Im}(P) = V$, then P must be the identity map on V (i.e. $P(v) = v$ for all $v \in V$).

(c) Give an example of a projection $P : \mathbb{C}^2 \to \mathbb{C}^2$ such that

$$\mathrm{Im}(P) = \left\langle \begin{bmatrix} s_8 \\ s_7 \end{bmatrix}, \begin{bmatrix} s_6 \\ s_5 \end{bmatrix}, \begin{bmatrix} s_4 \\ s_3 \end{bmatrix}, \begin{bmatrix} s_2 \\ s_1 \end{bmatrix} \right\rangle .$$

(d) Does there exist a $(s_5+2) \times (s_5+2)$ matrix A with characteristic polynomial $c_A(x) = (x - s_7)^{s_5+2}$ and minimal polynomial $m_A(x) = x - s_8$? Justify your answer.

3 This question asks you for two different proofs of the same fact.

An $n \times n$ matrix A is said to be *upper triangular* if all the entries below the main diagonal are zero; that is, if $A_{ij} = 0$ whenever $i > j$. Prove that the determinant of an upper triangular matrix is equal to the product of the diagonal entries

(a) by using the rules for a determinant function (Definition 4.1 in the notes); and

(b) by using the cofactor expansion (Theorem 4.6).

4 State whether the following are True or False, giving brief reasons.

(a) If A is a square matrix whose entries lie in a field K, then $\det(A^3) \in K$.

(b) For every field K and every $c \in K$, there exists a 4×4 matrix A, with entries in K, such that $\det(A) = c$.

(c) $\det(A+B) = \det(A) + \det(B)$ for all $n \times n$ matrices A and B.

5 Let A be the matrix over $\mathbb{R}$ given by

$$A = \begin{bmatrix} s_4 & s_2 & s_1 \\ s_7 & s_5 & s_3 \\ s_9 & s_8 & s_6 \end{bmatrix} .$$

(a) Compute all 9 cofactors of A, and use these to compute the adjugate $\mathrm{Adj}(A)$.

(b) Use the cofactor expansion (along the row or column of your choice) to compute $\det(A)$.

(c) Use (b) to state, with reason, whether or not A is an invertible matrix. If A is invertible then use (a) and (b) to compute A^{-1}.

(d) Compute the characteristic polynomial c_A of A, expressing it in the form $c_A(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta$.

(e) Compute A^2 and A^3, and verify directly (i.e. without quoting the Cayley-Hamilton Theorem) that $c_A(A) = 0$.

6 Let V be the vector space (over the field $\mathbb{R}$) consisting of all infinite sequences $(x_1, x_2, x_3, x_4, \ldots)$, where $x_i \in \mathbb{R}$ for all $i \geq 1$.

(a) Is V finite-dimensional? If so, what is its dimension?

(b) Write down a 1-dimensional subspace of V which contains the sequence

$$(s_1, s_2, s_3, s_4, 0, 0, 0, 0, 0, 0, 0, 0 \ldots).$$

(c) Give an example of a 2-dimensional subspace of V.

(d) Prove that $U = \{(x_1, x_2, x_3, x_4, \ldots) \in V : s_2 x_2 + s_3 x_3 = 0\}$ is a subspace of V.

(e) Is $W = \{(x_1, x_2, x_3, x_4, \ldots) \in V : s_7 x_7 + s_8 x_8 = s_9\}$ a subspace of V? Justify your answer.

7 (a) Let $S : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear map given by

$$S\left(\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+z \\ s_2(t+y) \\ s_3(x+z) \\ s_4(t+y) \end{bmatrix}.$$

Find the matrix A which represents S with respect to the standard basis for $\mathbb{R}^4$, and find a basis for the kernel of S.

(b) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear map given by

$$T\left(\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+z \\ t+y \\ x+z \\ t+y \end{bmatrix}.$$

Find the matrix B which represents T with respect to the standard basis for $\mathbb{R}^4$, and find the eigenvalues of T.

(c) Find a basis of $\mathbb{R}^4$ which consists of eigenvectors of T, and find an invertible matrix P such that $P^{-1}BP$ is a diagonal matrix.

8 Let A be the real matrix given by

$$A = \begin{bmatrix} -3 & 8 & 2 \\ -2 & 5 & 1 \\ -2 & 2 & 4 \end{bmatrix}.$$

(a) Compute the characteristic polynomial $c_A(x)$.

(b) Determine the minimal polynomial $m_A(x)$, taking care to justify your answer.

(c) Find the eigenvalues and eigenvectors of A.

(d) Find an invertible matrix Q such that $Q^{-1}AQ$ is a diagonal matrix.

(e) Find projection matrices P_1, P_2, P_3 such that

$$A = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3,$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of A.

9 (a) Determine the characteristic and minimal polynomials of the real matrix

$$A = \begin{bmatrix} s_1 & s_1 - 4 & -2s_2 \\ s_1 + 1 & s_1 + 4 & s_2 \\ s_1 & s_1 & s_3 \end{bmatrix},$$

taking care to justify your answer.

(b) For each positive integer n, what is the rank of the $n \times n$ matrix with every entry equal to 1? Justify your answer.

(c) For each positive integer n, determine the minimal polynomial of the $n \times n$ matrix with every entry equal to 1.

(d) For which values of n is the matrix in (c) diagonalisable? Justify your answer.