Queen Mary University of London

MTH6140 **Linear Algebra II**

Assignment 3 **Solutions**

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1 (a) [20%] If $w \in \mathrm{Im}(T)$ then $w = T(v)$ for some $v \in V$, so $T(w) = T(T(v)) = T^2(v) = 0$, so $w \in \ker(T)$. So we have shown that $\mathrm{Im}(T) \subset \ker(T)$.

(b) [20%] Yes, this is obvious: $\mathrm{Im}(T)$ and $\ker(T)$ are known to be subspaces of V (see Proposition 3.1 in the lecture notes), so in particular they are vector spaces in their own right. From part (a) we know that $\mathrm{Im}(T)$ is a subset of $\ker(T)$. So $\mathrm{Im}(T)$ is a subset of $\ker(T)$, and is a vector space in its own right; therefore it is a subspace of $\ker(T)$.

An alternative proof is to note, by (a), that $\mathrm{Im}(T)$ is a subset of $\ker(T)$, and then check directly that $\mathrm{Im}(T)$ satisfies the subspace rules (the proof of this is almost identical to the proof in Proposition 3.1 in the notes).

(c) [20%] From (b) we know that the dimension of $\mathrm{Im}(T)$ is no greater than the dimension of $\ker(T)$; in other words, $\mathrm{rank}(T) \leq \mathrm{nul}(T)$. Combining this inequality with the Rank-Nullity Theorem gives

$$2\,\mathrm{rank}(T) \leq \mathrm{rank}(T) + \mathrm{nul}(T) = \dim(V) = s_2 + s_3\,,$$

so $\mathrm{rank}(T) \leq (s_2 + s_3)/2$, as required.

2 [40%] To prove that v and w are linearly independent we must show that if $c_1 v + c_2 w = 0$ then necessarily $c_1 = c_2 = 0$.

If $c_1 v + c_2 w = 0$ then applying T to both sides, and using linearity of T, we get $c_1 T(v) + c_2 T(w) = T(0) = 0$, and therefore

$$c_1(s_2 + s_9)v - c_2(s_2 + s_9)w = 0\,.$$

But multiplying the equation $c_1 v + c_2 w = 0$ by $s_2 + s_9$ gives

$$c_1(s_2 + s_9)v + c_2(s_2 + s_9)w = 0\,.$$

Adding together the above two equations gives

$$2c_1(s_2 + s_9)v = 0\,,$$

but v is a non-zero vector, and your $s_2 + s_9 \neq 0$, therefore $c_1 = 0$.

Plugging $c_1 = 0$ back into one of the previous equations gives

$$c_2(s_2 + s_9)w = 0\,,$$

but since $w \neq 0$ and $s_2 + s_9 \neq 0$ we get $c_2 = 0$.
So we have proved that $c_1 = 0 = c_2$, hence v and w are linearly independent, as required.

3 (a) The map T is linear. To prove this, first note that if $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$ belong to $\mathbb{R}^4$, then

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+x_3 \\ x_4+y_4 \end{bmatrix}\right) = s_1(x_1+y_1) + \sqrt{s_2}(x_2+y_2) - s_5(x_3+y_3) + s_7^2(x_4+y_4)\,.$$

But rearranging the righthand side of this equation gives:

$$\left(s_1x_1 + \sqrt{s_2}x_2 - s_5x_3 + s_7^2x_4\right) + \left(s_1y_1 + \sqrt{s_2}y_2 - s_5y_3 + s_7^2y_4\right),$$

and this is equal to

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) + T\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}\right).$$

So we have proved that

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) + T\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}\right),$$

which is one half of the definition of linearity of T. To check the other half of the definition of linearity, suppose that $\alpha \in \mathbb{R}$. Then

$$T\left(\alpha\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \\ \alpha x_4 \end{bmatrix}\right) = s_1(\alpha x_1) + \sqrt{s_2}(\alpha x_2) - s_5(\alpha x_3) + s_7^2(\alpha x_4)\,.$$

But this righthand side simplifies to the expression

$$\alpha(s_1x_1 + \sqrt{s_2}x_2 - s_5x_3 + s_7^2x_4) = \alpha T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right).$$

So we have proved that

$$T\left(\alpha\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\right)=\alpha T\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\right).$$

This completes the proof that T is linear.

(b) No. If such a map T existed then $\mathrm{rank}(T)+\mathrm{nul}(T)=\dim(\mathbb{R}^{2s_9+3})=2s_9+3$ by Theorem 3.2, so if $\mathrm{rank}(T)=\mathrm{nul}(T)$ we obtain $2\,\mathrm{rank}(T)=2s_9+3$. But this equation is impossible, because $\mathrm{rank}(T)$ is an integer and $2s_9+3$ is odd.

(c) Yes if $s_2=7$ or 4, but otherwise No.

If your s_2 does *not* equal 7 or 4 then the condition $2\,\mathrm{rank}(T)=\mathrm{nul}(T)$, together with the identity $\mathrm{rank}(T)+\mathrm{nul}(T)=\dim(\mathbb{R}^{s_2-1})=s_2-1$ from Theorem 3.2, gives $3\,\mathrm{rank}(T)=s_2-1$, which is impossible because $\mathrm{rank}(T)$ is an integer.

If your $s_2=7$ then you simply have to give an example of such a T. For example the map $T:\mathbb{R}^6\to\mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix}\right)=\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

(represented by the matrix $\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\end{bmatrix}$) has rank 2 and nullity 4. However there are many other valid examples!

If your $s_2=4$ then again you simply have to give an example of such a T. For example the map $T:\mathbb{R}^3\to\mathbb{R}^2$ defined by $T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\begin{bmatrix}x_1\\0\end{bmatrix}$ has rank 1 and nullity 2. Again, there are many other valid examples!

(d) Obviously there are many possible answers to this question. The best idea is probably to choose three real numbers a, b, c 'at random' (so that it's very unlikely anyone else will have selected them), then define U to be the span of the vector $\begin{bmatrix}a\\b\\c\end{bmatrix}$.

For example I might choose $a=-2745603$, $b=\pi^2/77$, and $c=13/4$, then define

$$U=\left\langle\begin{bmatrix}-2745603\\\pi^2/77\\13/4\end{bmatrix}\right\rangle=\left\{t\begin{bmatrix}-2745603\\\pi^2/77\\13/4\end{bmatrix}:t\in\mathbb{R}\right\}.$$

(Recall that, geometrically, every 1-dimensional subspace is a line through the origin. More precisely, the span of the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is the line through the origin which also passes through the point $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$).

4 (a) V is finite-dimensional. Its dimension is 14.

(b) $\langle s_1v_1, s_2v_3, s_3v_4, s_5v_7\rangle$ denotes the set of all linear combinations of the vectors s_1v_1, s_2v_3, s_3v_4, and s_5v_7. In other words,

$$\langle s_1v_1, s_2v_3, s_3v_4, s_5v_7\rangle = \{a_1s_1v_1 + a_2s_2v_3 + a_3s_3v_4 + a_4s_5v_7 : a_1, a_2, a_3, a_4 \in \mathbb{R}\}\ .$$

(c) Yes, $\langle s_1v_1, s_2v_3, s_3v_4, s_5v_7\rangle$ is finite-dimensional.
Its dimension depends on your particular student digits s_1, s_2, s_3, s_5. For everyone $s_1 = 0$, so $s_1v_1 = 0$, so $\langle s_1v_1, s_2v_3, s_3v_4, s_5v_7\rangle = \langle s_2v_3, s_3v_4, s_5v_7\rangle$, and therefore the dimension is ≤ 3.

In general the dimension of $\langle s_1v_1, s_2v_3, s_3v_4, s_5v_7\rangle$ is equal to $3-r$, where r denotes the number of 0's among the three digits s_2, s_3 and s_5.

(d) Yes, $\langle s_2v_3, s_3v_4, s_5v_7\rangle$ is finite-dimensional.

In fact, as noted in (c) above, this subspace is equal to $\langle s_1v_1, s_2v_3, s_3v_4, s_5v_7\rangle$, because $s_1 = 0$. Therefore, as before, its dimension equals $3-r$, where r denotes the number of 0's among the three digits s_2, s_3 and s_5.

(e) Yes, $\langle s_2v_3 + s_3v_4 + s_5v_7\rangle$ is finite-dimensional. It is the span of the single vector $s_2v_3 + s_3v_4 + s_5v_7$, so its dimension is equal to either 0 or 1.

For most of you its dimension equals 1. However if $s_2 = s_3 = s_5 = 0$ then $s_2v_3 + s_3v_4 + s_5v_7$ is the zero vector, in which case $\langle s_2v_3 + s_3v_4 + s_5v_7\rangle$ has dimension 0.

5 (a) The answer is either 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, or 10, depending on your student number.
For example if your number is 063827095 then $s_1+s_2+s_3+s_4+s_5+s_6+s_7+s_8+s_9$ is equal to

$$0+6+3+8+2+7+1+9+5 \pmod{11} = 31 \pmod{11} = 9+2\cdot 11 \pmod{11} = 9\,.$$

(b) Again the answer is either 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, or 10, depending on your student number. In particular, the answer is 0 if any of your student digits, other than s_1, are zero.

For example if your number is 063827195 then $s_2\,s_3\,s_4\,s_5\,s_6\,s_7\,s_8\,s_9$ is equal to

$$6\cdot 3\cdot 8\cdot 2\cdot 7\cdot 1\cdot 9\cdot 5 \pmod{11} = 90720 \pmod{11} = 3+8247\cdot 11 \pmod{11} = 3\,.$$

(c) $\mathbb{F}_{11}^{s_2}$ has dimension equal to s_2. For example if your number is 063827195 then $\mathbb{F}_{11}^{s_2}$ has dimension 6.

(d) There are 11^{s_2} vectors in $\mathbb{F}_{11}^{s_2}$. So in your case the number of vectors in $\mathbb{F}_{11}^{s_2}$ will be either $11^4 = 14641$, or $11^5 = 161051$, or $11^6 = 1771561$, or $11^7 = 19487171$, depending on whether your second student digit is 4, 5, 6, or 7. For example if your number is 063827195 then there are 1771561 vectors in $\mathbb{F}_{11}^{s_2}$.

(e) There are many ways of choosing a 1-dimensional subspace of $\mathbb{F}_{11}^{s_2}$, and all of these contain precisely 11 vectors.

Each 1-dimensional subspace U is of the form $U = \langle v \rangle$, where v is some non-zero member of $\mathbb{F}_{11}^{s_2}$. For example if your number is 063827195 then you might choose

$$U = \left\{ \begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}3\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}4\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}5\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}6\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}7\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}8\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}9\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}10\\0\\0\\0\\0\\0\end{bmatrix} \right\},$$

or you might instead choose

$$U = \left\{ \begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\3\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\\6\\0\\0\end{bmatrix}, \begin{bmatrix}3\\0\\0\\9\\0\\0\end{bmatrix}, \begin{bmatrix}4\\0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}5\\0\\0\\4\\0\\0\end{bmatrix}, \begin{bmatrix}6\\0\\0\\7\\0\\0\end{bmatrix}, \begin{bmatrix}7\\0\\0\\10\\0\\0\end{bmatrix}, \begin{bmatrix}8\\0\\0\\2\\0\\0\end{bmatrix}, \begin{bmatrix}9\\0\\0\\5\\0\\0\end{bmatrix}, \begin{bmatrix}10\\0\\0\\8\\0\\0\end{bmatrix} \right\}, \text{ etc.}$$

6 (a) $T\left(\begin{bmatrix}2\\-1\\0.25\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}0\\s_9\end{bmatrix}$. To see this we use linearity of T:

$$T\left(\begin{bmatrix}2\\-1\\0.25\end{bmatrix}\right) = T\left(2\begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix} + \frac{1}{4}\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = 2T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) - T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) + \frac{1}{4}T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\s_9/2\end{bmatrix}.$$

(b) The matrix representing T is

$$\begin{bmatrix}s_1 & s_8 & 4s_8\\ s_2 & 2s_2 & 2s_9\end{bmatrix}.$$

(c) Rank $= 2$ if $s_8 \neq 0$. If $s_8 = 0$ then rank $= 1$. (Note that everyone's student number has $s_1 = 0$ and $s_2 \neq 0$).

(d) $S\left(\begin{bmatrix}s_4\\-s_7\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\end{bmatrix}$, since the map S is identically zero (i.e. $S\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\end{bmatrix}$ for all $\begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2$).

7 (a) $\text{rank}(T) = 2$ or 3, depending on your student number. More precisely, $\text{rank}(T) = 2$ if your s_3 equals either 0 or $s_2 - 3$; otherwise $\text{rank}(T) = 3$. *[Note that this answer exploits some features of your student numbers: in particular you all have that $s_1 = 0$ and $s_2 - 3 \neq s_1$].*

(b) $\text{nul}(T) = 0$ or 1. More precisely, $\text{nul}(T) = 1$ if your s_3 equals either 0 or $s_2 - 3$; otherwise $\text{nul}(T) = 0$. This is because, by Theorem 3.2, $\text{nul}(T) = \dim(\mathbb{R}^3) - \text{rank}(T) = 3 - \text{rank}(T)$.

8 (a) Two matrices A and B, of the same size, are *equivalent* if there exist invertible square matrices P and Q such that $B = PAQ$.

[Acceptable alternative definitions of A and B being equivalent are: (i) A and B have the same rank, or (ii) A and B can be reduced, by a sequence of elementary row and column operations, to the same canonical form for equivalence, or (iii) A can be transformed into B via a sequence of elementary row and column operations].

(b) The size of the matrices depends on your student number. For example if your number is 063827195 then the two matrices are both of size 10×6.

An easy example is to choose A with $A_{11} = 1 = A_{12}$ and $A_{ij} = 0$ otherwise, and B with $B_{11} = 1$ and $B_{ij} = 0$ otherwise. These matrices are equivalent because if we replace the 2nd column of A by the difference of its first and second columns, then we obtain B.

Alternatively, you might choose A and $B = PAQ$ as in (d) below (assuming your working there is correct)!

(c) The answer here depends on whether your $s_4 = 0$ or not.

If $s_4 = 0$ then the canonical form for equivalence is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so rank $= 1$. To see this, swap the 1st and 2nd rows, then subtract s_2 copies of the first column from the second column, and the matrix is already in canonical form. We have

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & -s_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If $s_4 \neq 0$ then the canonical form for equivalence is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so rank $= 2$. To see this, swap the 1st and 2nd rows, next subtract s_2 copies of the first column from the second column, then swap the 2nd and 3rd rows, then swap the 2nd and 3rd columns, and finally multiply the 2nd row by s_4^{-1} to obtain the canonical form. We have

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & s_4^{-1} \\ 1 & 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & -s_2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(d) Since $T(v) = Av$ where A is as in (d), the rank of T equals the rank of A. So $\text{rank}(T) = 1$ if $s_4 = 0$, and $\text{rank}(T) = 2$ if $s_4 \neq 0$.