MTH6140 **Linear Algebra II**

Assignment 3 **For handing in on Monday 15th November**

Questions 1 and 2 only *will be marked for credit. Write your name and student number at the top of your solution before handing it in. Post the solution in the* ***YELLOW*** *post-box in the* ***BASEMENT*** *of the Maths building before 15:00 on Monday.*

Let $s_1\, s_2\, s_3\, s_4\, s_5\, s_6\, s_7\, s_8\, s_9$ **denote your student number (i.e. the digits of your student number are, in order,** s_1**,** s_2**,** s_3**,** s_4**,** s_5**,** s_6**,** s_7**,** s_8**,** s_9**).**

1 Let $T : V \to V$ be a linear map, where V is a vector space of dimension $s_2 + s_3$. Suppose that $T^2 = O$ (i.e. the composition $T^2 = T \circ T$ is the map which sends every vector to the zero vector).

(a) Show that $\mathrm{Im}(T) \subset \ker(T)$.

(b) Is $\mathrm{Im}(T)$ a subspace of $\ker(T)$?

(c) Use the Rank-Nullity Theorem to show that $\mathrm{rank}(T) \leq (s_2 + s_3)/2$.

2 Let $T : V \to V$ be a linear map, and suppose there are non-zero vectors $v, w \in V$ such that $T(v) = (s_2 + s_9)v$ and $T(w) = -(s_2 + s_9)w$.
Show that v and w are linearly independent.

3 (a) Suppose that the map $T : \mathbb{R}^4 \to \mathbb{R}$ is defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = s_1 x_1 + \sqrt{s_2}\, x_2 - s_5 x_3 + s_7^2 x_4\,.$$

Either prove, or disprove, that T is linear.

(b) Does there exist a linear map $T : \mathbb{R}^{2s_9+3} \to \mathbb{R}^{s_5+1}$ such that $\operatorname{rank}(T) = \operatorname{nul}(T)$? Justify your answer.

(c) Does there exist a linear map $T : \mathbb{R}^{s_2-1} \to \mathbb{R}^{s_1+2}$ such that $2\operatorname{rank}(T) = \operatorname{nul}(T)$? Justify your answer.

(d) Give an example of a 1-dimensional subspace U of $\mathbb{R}^3$ which is *different* from the example given by all other students answering this question.

4 Let $(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14})$ be a basis for a vector space V over the field $\mathbb{R}$.

(a) Is V finite-dimensional? If so, what is its dimension?

(b) What does the notation $\langle s_1v_1, s_2v_3, s_3v_4, s_5v_7\rangle$ mean?

(c) Is $\langle s_1v_1, s_2v_3, s_3v_4, s_5v_7\rangle$ finite-dimensional? If so, what is its dimension?

(d) Is $\langle s_2v_3, s_3v_4, s_5v_7\rangle$ finite-dimensional? If so, what is its dimension?

(e) Is $\langle s_2v_3 + s_3v_4 + s_5v_7\rangle$ finite-dimensional? If so, what is its dimension?

5 Let $\mathbb{F}_{11}$ denote the field of integers modulo 11, and consider the digits s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9 in your student number to be elements of $\mathbb{F}_{11}$.

(a) Calculate $s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + s_7 + s_8 + s_9$ as an element of $\mathbb{F}_{11}$.

(b) Calculate the product $s_2\,s_3\,s_4\,s_5\,s_6\,s_7\,s_8\,s_9$ as an element of $\mathbb{F}_{11}$.

(c) What is the dimension of the vector space $\mathbb{F}_{11}^{s_2}$?

(d) How many vectors belong to $\mathbb{F}_{11}^{s_2}$?

(e) Give an example of a 1-dimensional subspace of $\mathbb{F}_{11}^{s_2}$. How many vectors belong to this subspace?

6 Suppose that the map $T : \mathbb{R}^3 \to \mathbb{R}^2$ is linear, and satisfies

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}s_1\\s_2\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}s_8\\2s_2\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}4s_8\\2s_9\end{bmatrix}.$$

(a) What is $T\left(\begin{bmatrix}2\\-1\\0.25\end{bmatrix}\right)$? Justify your answer.

(b) What is the matrix representing T relative to the bases $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$ for $\mathbb{R}^3$ and $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$ for $\mathbb{R}^2$?

(c) What is the rank of the matrix from part (b)? Briefly justify your answer.

(d) If $S : \mathbb{R}^2 \to \mathbb{R}^3$ is linear, with $S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, then compute $S\left(\begin{bmatrix} s_4 \\ -s_7 \end{bmatrix} \right)$.

7 Define

$$v_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

and define the linear map $T : \mathbb{R}^3 \to \mathbb{R}^6$ by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = x_1 v_{s_1} + x_2 v_{s_2 - 3} + x_3 v_{s_3}.$$

(a) Compute $\mathrm{rank}(T)$.

(b) Compute $\mathrm{nul}(T)$.

8 (a) Explain carefully what it means for two matrices to be *equivalent*.

(b) Give an example of two different $(s_8 + 1) \times (s_9 + 1)$ matrices which are equivalent, justifying *very briefly* your answer.

(c) Consider the matrix

$$A = \begin{bmatrix} s_1 & 0 & 0 \\ 1 & s_2 & 0 \\ 0 & 0 & s_4 \end{bmatrix}$$

over the field $\mathbb{R}$.

What is its canonical form for equivalence, and what is its rank? Find invertible matrices P and Q such that PAQ is in the canonical form for equivalence.

(d) Determine the rank of the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} s_1 x \\ x + s_2 y \\ s_4 z \end{bmatrix}.$$