MTH6140 **Linear Algebra II**

Assignment 2 **Solutions**

---

1 *(a) If $s_3 \neq 0$ then the rank is 2 and the canonical form for equivalence is* $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

If $s_3 = 0$ then the rank is 1 and the canonical form for equivalence is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(Note that the above answers rely on the fact that you all have $s_1 = 0$ and $s_2 \neq 0$).

(b) If s_2 and s_3 are **both odd** *then the rank is 2, and the canonical form for equivalence is* $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

If s_2, s_3 and s_4 are **all even** *then the rank is 0 and the canonical form is* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Otherwise the rank is 1, and the canonical form for equivalence is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(c) If s_2 and s_3 are **both not a multiple of 3** *then the rank is 2, and the canonical form for equivalence is* $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

If s_2, s_3 and s_4 are **all multiples of 3** *then the rank is 0 and the canonical form for equivalence is* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Otherwise the rank is 1, and the canonical form for equivalence is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

2 (a) The equation

$$A = \tfrac{1}{2}(A + A^\top) + \tfrac{1}{2}(A - A^\top)$$

shows that every matrix is the sum of a symmetric and a skew-symmetric matrix. ($A + A^\top$ is symmetric because

$$(A + A^\top)^\top = A^\top + (A^\top)^\top = A^\top + A,$$

and similarly $A - A^\top$ is skew-symmetric.) Thus

$$M_n(\mathbb{R}) = S_n(\mathbb{R}) + A_n(\mathbb{R}).$$

Also, suppose that A is both symmetric and skew-symmetric. Then

$$A = A^\top = -A,$$

so $2A = O$, whence $A = O$. This shows that $S_n(\mathbb{R}) \cap A_n(\mathbb{R}) = \{O\}$, and hence

$$M_n(\mathbb{R}) = S_n(\mathbb{R}) \oplus A_n(\mathbb{R}).$$

(b) **Yes** if the prime $p \geq 3$, using the same proof as in (a) (with $\mathbb{R}$ replaced by $\mathbb{F}_p$).

No if $p = 2$: in this case $-x = x$ for all x, so the definitions of "symmetric" and "skew-symmetric" coincide, so $S_n(\mathbb{F}_2) \cap A_n(\mathbb{F}_2) = S_n(\mathbb{F}_2) \neq \{O\}$.

3 (a) No, it is not. Although $\mathbb{R}^3$ is a sub*set* of $\mathbb{C}^3$, it is not a sub*space*. Note that $\mathbb{C}^3$ is (by definition) a vector space over $\mathbb{C}$, but for example $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3$ yet $iv = \begin{bmatrix} i \\ i \\ i \end{bmatrix} \notin \mathbb{R}^3$. [Here i denotes the square root of -1].

(b) Yes, it is true that $s_5u - s_9u' \in U$. This follows from the definition of a subspace: $s_5 \in \mathbb{R}$ so $s_5u \in U$, and $-s_9 \in \mathbb{R}$ so $-s_9u' \in U$, therefore $s_5u - s_9u' = s_5u + (-s_9u') \in U$.

(c) It is a subspace if $s_4 = 0$ (to show this, just check it satisfies the definition of a subspace), but is not a subspace if $s_4 \neq 0$ (one way of showing this is to find a vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ which lies in U, but such that some scalar multiple, for example $\begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$, does not lie in U).

(d) You all have $s_1 = 0$, so $A^2 = \begin{bmatrix} 0 & 0 & s_3s_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. This clearly has rank 0 if $s_3 = 0$ or $s_4 = 0$, and has rank 1 otherwise.

(e) $p = 11$ for all of you (at least I *think* this is true - everyone has a digit ≥ 7 in their student number, don't they?), so the field is $\mathbb{F}_{11}$. Now the second column in A is equal to twice the first column (since $12 \equiv 1$ modulo 11), so the (column) rank of A equals 1. The canonical form for equivalence is therefore $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

4 (a) Subtract the first row from the second, add the first row to the third, then multiply the new second row by -1 and subtract four times this row from the third, to get the matrix

$$B = \begin{bmatrix} 1 & 2 & 4 & -1 & 5 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first two rows clearly form a basis for the row space.

(b) The rank is 2, since there is a basis with two elements.

(c) The column rank is equal to the row rank and so is also equal to 2. By inspection, the first and third columns are linearly independent, so they form a basis.

(d) By step (a), we have $PA = B$, where P is obtained by performing the same elementary row operations on the 3×3 identity matrix I_3:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -3 & 4 & 1 \end{bmatrix}.$$

Now B can be brought to the canonical form

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

by subtracting 2, 4, -1 and 5 times the first column from the second, third, fourth and fifth columns, and twice the third column from the fifth, and then swapping the second and third columns; so $C = BQ$ (whence $C = PAQ$), where Q is obtained by performing the same column operations on I_5:

$$Q = \begin{bmatrix} 1 & -4 & -2 & 1 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Remark: P and Q can also be found by multiplying elementary matrices, if desired; but the above method is simpler.

5 This can be solved using elementary row and column operations to reduce A to a matrix in canonical form for equivalence (as in the lecture notes), then counting the number of 1's in this matrix.

The rank of A is equal to the rank of $A^\top$ (since e.g. the row rank of A clearly equals the column rank of $A^\top$).

Depending on your student number, this rank will be either 1, 2 or 3. For most of you the rank will be 3.

6 There are $2^4 = 16$ such matrices.

One matrix has rank 0, namely $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

9 matrices have rank 1, namely $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, and $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

6 matrices have rank 2: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

None of the matrices have rank 3 or 4 (in general the rank of an $n \times n$ matrix is $\leq n$).

7 (a) $\langle v_1, v_2, v_3, v_4, v_5 \rangle := \{a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + a_5 v_5 : a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}\}$.

(b) We now prove that $\langle v_1, v_2, v_3, v_4, v_5 \rangle = \langle v_1 - 3v_4, v_2, v_3, v_4, v_5 \rangle$.
If $v \in \langle v_1, v_2, v_3, v_4, v_5 \rangle$ then

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + a_5 v_5$$

for some $a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}$. But we can re-write this as

$$v = a_1(v_1 - 3v_4) + a_2 v_2 + a_3 v_3 + (3a_1 + a_4)v_4 + a_5 v_5 \in \langle v_1 - 3v_4, v_2, v_3, v_4, v_5 \rangle,$$

hence $\langle v_1, v_2, v_3, v_4, v_5 \rangle \subset \langle v_1 - 3v_4, v_2, v_3, v_4, v_5 \rangle$.

Similarly, if

$$w = b_1(v_1 - 3v_4) + b_2 v_2 + b_3 v_3 + b_4 v_4 + b_5 v_5 \in \langle v_1 - 3v_4, v_2, v_3, v_4, v_5 \rangle$$

then

$$w = b_1 v_1 + b_2 v_2 + b_3 v_3 + (b_4 - 3b_1)v_4 + b_5 v_5 \in \langle v_1, v_2, v_3, v_4, v_5 \rangle,$$

hence $\langle v_1 - 3v_4, v_2, v_3, v_4, v_5 \rangle \subset \langle v_1, v_2, v_3, v_4, v_5 \rangle$, and the proof is complete.

(c) This holds if and only if $v_1 = cv_2$ for some non-zero scalar $c \in \mathbb{R}$.

8 None of these sets are subspaces of $\mathbb{R}^3$.

In fact the only subspaces of $\mathbb{R}^3$ are:

(i) The trivial subspace $\{0\}$. [This has dimension equal to 0].

(ii) Any line through the origin, i.e. any set of the form $\left\{ t \begin{bmatrix} x \\ y \\ z \end{bmatrix} : t \in \mathbb{R} \right\}$ for some non-zero vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. [Such subspaces have dimension equal to 1].

(iii) Any plane through the origin, i.e. any set of the form $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : ax + by + cz = 0 \right\}$ for some non-zero vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$. [Such subspaces have dimension equal to 2].

(iv) $\mathbb{R}^3$ itself. [This has dimension equal to 3].