MTH6140 Linear Algebra II

Assignment 2 For handing in on Monday 25th October

> **Questions 1 and 2 only** *will be marked for credit. Write your name and student number at the top of your solution before handing it in. Post the solution in the* ***YELLOW*** *post-box in the* ***BASEMENT*** *of the Maths building before 15:00 on Monday.*

Let $s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9$ **denote your student number (i.e. the digits of your student number are, in order,** $s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9$**).**

1 (20% credit for each of the three parts)

Find the rank, and the canonical form for equivalence, of the matrix $\begin{bmatrix} s_1 & s_3 \\ s_2 & s_4 \end{bmatrix}$

(a) over $\mathbb{R}$,
(b) over $\mathbb{F}_2$,
(c) over $\mathbb{F}_3$.

2 *(20% credit for each of the two parts)*

An $n \times n$ *matrix* A *is said to be* symmetric *if* $A^\top = A$, *and is said to be* skew-symmetric *if* $A^\top = -A$ *(where* $\top$ *denotes transpose). Let* $S_n(K)$ *be the set of* $n \times n$ *symmetric matrices over a field* K, *and* $A_n(K)$ *the set of* $n \times n$ *skew-symmetric matrices. You may assume that both these sets are subspaces of the set* $M_n(K)$ *of all* $n \times n$ *matrices.*

(a) In the case when $K = \mathbb{R}$, *show that*

$$M_n(\mathbb{R}) = S_n(\mathbb{R}) \oplus A_n(\mathbb{R})\,.$$

(b) Does the equality

$$M_n(K) = S_n(K) \oplus A_n(K)$$

hold in the case when $K = \mathbb{F}_p$, *for* p *a prime? Briefly justify your answer.*

3 (a) Is $\mathbb{R}^3$ a subspace of $\mathbb{C}^3$? Justify your answer.

(b) Let U be a subspace of $\mathbb{R}^{s_4+2}$, and let $u, u' \in U$. Does it necessarily follow that $s_5 u - s_9 u' \in U$? Justify your answer.

(c) Is the set

$$U = \left\{ \begin{bmatrix} t^{1+s_4} \\ (s_7 + s_9)t \\ (s_5 - 3s_6)t \end{bmatrix} : t \in \mathbb{R} \right\}$$

a subspace of $\mathbb{R}^3$? Justify your answer.

(d) If $A = \begin{bmatrix} s_1 & s_3 & s_2 \\ 0 & 0 & s_4 \\ 0 & 0 & s_1 \end{bmatrix}$ is a matrix over $\mathbb{Q}$, what is the rank of A^2? Justify your answer.

(e) Let p be the smallest prime number which is strictly larger than all of the digits s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9 in your student number, and let $A = \begin{bmatrix} 6 & 3 \\ 1 & 6 \end{bmatrix}$ be a matrix over the field $K = \mathbb{F}_p$ (the field of integers mod p). What is the rank of A, and what is its canonical form for equivalence? Briefly justify your answer.

4 Let

$$A = \begin{bmatrix} 1 & 2 & 4 & -1 & 5 \\ 1 & 2 & 3 & -1 & 3 \\ -1 & -2 & 0 & 1 & 3 \end{bmatrix}.$$

(a) Find a basis for the row space of A.

(b) What is the rank of A?

(c) Find a basis for the column space of A.

(d) Find invertible matrices P and Q such that PAQ is in the canonical form for equivalence.

5 Let

$$A = \begin{bmatrix} s_1 & s_2 & s_3 \\ s_4 & s_5 & s_6 \\ s_7 & s_8 & s_9 \end{bmatrix}$$

be a matrix over $\mathbb{R}$.

What is the rank of A?
What is the rank of the transpose $A^\top$?
Justify your answer in both cases.

6 How many 2×2 matrices are there over $K = \mathbb{F}_2$ (the field of integers mod 2)?

How many of these matrices have rank 0? How many have rank 1? How many have rank 2? How many have rank 3? How many have rank 4?

7 Let v_1, v_2, v_3, v_4, v_5 be vectors in a vector space V over $\mathbb{R}$.

(a) Give the definition of $\langle v_1, v_2, v_3, v_4, v_5 \rangle$.

(b) Prove that $\langle v_1, v_2, v_3, v_4, v_5 \rangle = \langle v_1 - 3v_4, v_2, v_3, v_4, v_5 \rangle$.

(c) Under what circumstances does the equality $\langle v_1 \rangle = \langle v_2 \rangle$ hold?

8 Which of the following are subspaces of $\mathbb{R}^3$? Give brief reasons.

(a) The unit sphere $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \right\}$,

(b) The unit ball $B = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1 \right\}$,

(c) The line segment $L = \left\{ \begin{bmatrix} t \\ 3t \\ -t \end{bmatrix} : t \in [0,6] \right\}$,

(d) The plane $P = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 3x - 7y + 4z = 1 \right\}$,

(e) The set of integer vectors $\mathbb{Z}^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x, y, z \in \mathbb{Z} \right\}$.