MTH6140 **Linear Algebra II**

Assignment 1 **Solutions**

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1 [For this marked question, each of the five parts is worth 20%].

(a) Of course there is no single correct answer to this question, but I recommend choosing the vector space $V = \mathbb{Q}^n$ where n is a natural number that other students are unlikely to choose. For example: $V = \mathbb{Q}^{385729998}$.

(b) I recommend choosing $V = \mathbb{C}^{s_2+s_3}$, and letting the basis consist of vectors v_i, for $1 \leq i \leq s_2+s_3$, where each v_i is defined to have the entry 1 as its ith component, and the entry 0 for all other components. (However, of course this is not the only correct answer).

So if, for example, you have $s_2 = 2$ and $s_3 = 1$, then choose $V = \mathbb{C}^3$, with basis consisting of the three vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(c) Once again, there is no single correct answer to this question, but you might choose $V = \mathbb{R}^2$ as your vector space. Since this space is 2-dimensional, we can choose two vectors (for example $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$) to form a basis, and declare these to be the first two vectors in our list. Then simply choose the remaining s_9+1 vectors in our list to be anything we like; the resulting list certainly spans $\mathbb{R}^2$, but is not a basis (since it is not linearly independent).

(d) Almost all of you will have student numbers where the 3 vectors *are* linearly independent, and a quick proof (which is not required in this question) is to calculate that the determinant

$$\det \begin{bmatrix} s_1 & is_4 & s_7 \\ is_2 & s_5 & s_8 \\ s_3 & s_6 & is_9 \end{bmatrix} = s_3s_4s_8i - s_3s_5s_7 + s_2s_6s_7i - s_1s_6s_8 + s_1s_5s_9i - s_2s_4s_9(-i)$$

is *non-zero*. Alternatively, use the definition of linear independence to show that the only complex numbers c_1, c_2, c_3 satisfying

$$c_1 \begin{bmatrix} s_1 \\ is_2 \\ s_3 \end{bmatrix} + c_2 \begin{bmatrix} is_4 \\ 2_5 \\ s_6 \end{bmatrix} + c_3 \begin{bmatrix} s_7 \\ s_8 \\ is_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

are $c_1 = c_2 = c_3 = 0$.

You may be among the small minority for whom the three vectors are *not* linearly independent - in this case the above determinant would be zero, and there exist non-zero c_1, c_2, c_3 satisfying the above equation.

(e) If your $s_7 = 0$ then U *is* a subspace. If your $s_7 \neq 0$ then U is *not* a subspace.

Tip: To get the answer, without formally testing U against the definition of subspace, remember the rule of thumb: parametrising variables (in this case the variable a) should appear with degree EXACTLY ONE if U is to be a subspace. Note that $s_7 = 0$ ensures $s_7 + 1 = 1$, so in the term $3a^{s_7+1} = 3a$ the variable a *does* appear with degree equal to one.

Justification of answer:

If your $s_7 \neq 0$ then to show U is not a subspace it would be sufficient to find some vector u which belongs to U but such that some multiple, for example, $2u$, does not belong to U. To generate this vector u you might choose $a = 1$, giving

$$u = \begin{bmatrix} 2s_2 - s_3 \\ 3 \\ s_6 + s_8 \end{bmatrix} \in U\,.$$

If your $s_7 = 0$ then to show U *is* a subspace you must check that for all $u, u' \in U$ and $c \in \mathbb{R}$, we have that $u + u' \in U$ and $cu \in U$.

2 (a) To show linear independence, suppose that for real numbers c_1, c_2 and c_3 we have:

$$c_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 9 \\ -11 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We would like to show that *necessarily* $c_1 = c_2 = c_3 = 0$. One way of seeing this is to write the above equation as

$$\begin{bmatrix} 3 & 4 & 9 \\ 2 & 1 & -11 \\ 1 & 5 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

so that

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 9 \\ 2 & 1 & -11 \\ 1 & 5 & -5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{227} \begin{bmatrix} 50 & 65 & -53 \\ -1 & -24 & 51 \\ 9 & -11 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and this vector equation tells us precisely that $c_1 = c_2 = c_3 = 0$, as required.

[Note that the above proof works because the matrix $A = \begin{bmatrix} 3 & 4 & 9 \\ 2 & 1 & -11 \\ 1 & 5 & -5 \end{bmatrix}$ **is *invertible*, which is equivalent to** $\det(A) \neq 0$**; full marks can also be scored by simply using this observation, rather than giving the full calculation above.]**

To show the vectors are spanning we have a choice of methods. One method is to recall (see e.g. Theorem 4.44 in Linear Algebra I) that if n vectors in an n-dimensional space are linearly independent then they are *automatically* spanning (where $n = 3$ in this case). Another method is to directly use the definition of spanning: we have to check that for any vector $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$, we can find scalars a, b, c such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} + c \begin{bmatrix} 9 \\ -11 \\ -5 \end{bmatrix}.$$

If we re-write the above vector equation as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 4 & 9 \\ 2 & 1 & -11 \\ 1 & 5 & -5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

then we see it has solution

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 & 4 & 9 \\ 2 & 1 & -11 \\ 1 & 5 & -5 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{227} \begin{bmatrix} 50 & 65 & -53 \\ -1 & -24 & 51 \\ 9 & -11 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

so $a = \frac{1}{227}(50x + 65y - 53z)$, $b = \frac{1}{227}(-x - 24y + 51z)$, $c = \frac{1}{227}(9x - 11y - 5z)$, so indeed the scalars a, b, c exist, as required.

(b) Almost all of you will have student number such that B' *is* a basis (so in particular is both linearly independent and spanning). You can use the same methods as in part (a) to prove this. Alternatively, simply check that the determinant

$$\det \begin{bmatrix} s_1 & s_4 & s_7 \\ s_2 & s_5 & s_8 \\ s_3 & s_6 & s_9 \end{bmatrix} = s_3s_4s_8 - s_3s_5s_7 + s_2s_6s_7 - s_1s_6s_8 + s_1s_5s_9 - s_2s_4s_9$$

is *non-zero*.

On the other hand if this determinant *does* equal zero for your student number, then your B' is *not* a basis (and in fact the vectors are neither linearly independent nor spanning, again by Theorem 4.44 in Linear Algebra I).

(c) Here the technique is essentially the same for each of the 3 vectors in B', and indeed is essentially the same as the calculation used in the 'spanning' part of (a) above. For example for the first vector $\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$, its *coordinates* with respect to the basis B are the scalars a, b, c such that

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = a \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} + c \begin{bmatrix} 9 \\ -11 \\ -5 \end{bmatrix} ,$$

and the corresponding *coordinate representation* is (a,b,c). By a computation analogous to that in (a) we see that the coordinate representation is

$$\frac{1}{227}\left(50s_1 + 65s_2 - 53s_3 \ , \ -s_1 - 24s_2 + 51s_3 \ , \ 9s_1 - 11s_2 - 5s_3\right) .$$

Similarly, the coordinate representation of $\begin{bmatrix} s_4 \\ s_5 \\ s_6 \end{bmatrix}$ (with respect to B) is

$$\frac{1}{227}\left(50s_4 + 65s_5 - 53s_6 \ , \ -s_4 - 24s_5 + 51s_6 \ , \ 9s_4 - 11s_5 - 5s_6\right) ,$$

and the coordinate representation of $\begin{bmatrix} s_7 \\ s_8 \\ s_9 \end{bmatrix}$ (with respect to B) is

$$\frac{1}{227}\left(50s_7 + 65s_8 - 53s_9 \ , \ -s_7 - 24s_8 + 51s_9 \ , \ 9s_7 - 11s_8 - 5s_9\right) ,$$

(d) Assuming your B' *is* a basis, the transition matrix $P = P_{B,B'}$ is defined (see Definition 1.6) to be the 3×3 matrix whose jth column is the coordinate representation (interpreted as a *column* vector) of the jth vector of B' relative to the basis B. Thus

$$P = \frac{1}{227} \begin{bmatrix} 50s_1 + 65s_2 - 53s_3 & 50s_4 + 65s_5 - 53s_6 & 50s_7 + 65s_8 - 53s_9 \\ -s_1 - 24s_2 + 51s_3 & -s_4 - 24s_5 + 51s_6 & -s_7 - 24s_8 + 51s_9 \\ 9s_1 - 11s_2 - 5s_3 & 9s_4 - 11s_5 - 5s_6 & 9s_7 - 11s_8 - 5s_9 \end{bmatrix} .$$

3 (a) True. The three vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}$$

are linearly independent, because if

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

for some $c_1, c_2, c_3 \in \mathbb{R}$ then $c_1 + c_2 + c_3 = 0$, and $c_2 + c_3 = 0$, and $7c_3 = 0$. Therefore $c_3 = 0$, hence $c_2 = -c_3 = 0$, and hence $c_1 = -c_2 - c_3 = 0$.

The list $B = (v_1, v_2, v_3)$ is also spanning. One proof of this is to note that three linearly independent vectors in (the 3-dimensional space) $\mathbb{R}^3$ automatically constitute a spanning set (by Proposition 1.8(b)).

(b) True. The three vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent, because if

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

for some $c_1, c_2, c_3 \in \mathbb{C}$ then $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$.

The list $B = (v_1, v_2, v_3)$ is also spanning, since three linearly independent vectors in (the 3-dimensional space) $\mathbb{C}^3$ automatically constitute a spanning set.

(c) False. The vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} i \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ i \end{bmatrix}$$

are spanning, but are not linearly independent, hence do not constitute a basis. To see that these vectors are not linearly independent, note (for example) that if $c_1 = -i$, $c_2 = 0$, $c_3 = 1$, $c_4 = 0$ then $c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$.

(d) False. For example $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in U$, but $2v = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \notin U$.

(e) True. If $u, u' \in U$ then

$$u = \begin{bmatrix} 0 \\ s \\ s+t \end{bmatrix}, \ u' = \begin{bmatrix} 0 \\ s' \\ s'+t' \end{bmatrix}$$

for some $s, s', t, t' \in \mathbb{R}$, so the sum

$$u + u' = \begin{bmatrix} 0 \\ s + s' \\ (s + s') + (t + t') \end{bmatrix}$$

does belong to U, and if $a \in \mathbb{R}$ then

$$au = \begin{bmatrix} 0 \\ as \\ as + at \end{bmatrix}$$

also belongs to U.

4 (a) False. Whenever the field K is infinite (e.g. if it equals $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Q}$) then every vector space V over K will contain *infinitely* many elements unless $V = \{0\}$.

(b) False. Whenever the field K is finite (e.g. if it equals $\mathbb{F}_p$, the field of integers mod p for some prime p) then every finite-dimensional vector space V over K (e.g. $V = K^n$ for some $n \geq 1$) will contain only *finitely* many elements.

(c) False. For example if $\{0\} \neq U_1 \subset U_2$ then $U_1 + U_2 = U_2$, so $\dim(U_1 + U_2) = \dim(U_2) < \dim(U_1) + \dim(U_2)$. In general $\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2)$ holds if and only if $U_1 \cap U_2 = \{0\}$.

(d) True. If $u_2 \in U_2$ then $-u_2 \in U_2$ as well, so $u_1 - u_2 \in U_1 + U_2$.

5 As usual with proofs, there are several approaches ... Here is one.

Let $\dim(U) = \dim(V) = n$, and choose a basis $(u_1, \ldots, u_n)$ for U. Then $u_1, \ldots, u_n$ are linearly independent vectors in V, a space of dimension n; so they form a basis for V. So in particular $u_1, \ldots, u_n$ is a spanning set; this means that every vector $v \in V$ is a linear combination of $u_1, \ldots, u_n$, and hence lies in U. So $V \subseteq U$. But we are given that $U \subseteq V$; so $U = V$.

6 Let $S = (v_1, \ldots, v_n)$ be spanning. If it is a basis, then we are done, so suppose not. This means that it is not linearly independent, so we have

$$c_1 v_1 + \cdots + c_n v_n = 0,$$

where $c_1, \ldots, c_n$ are not all zero. Select one of them which is not zero. Suppose that $c_n \neq 0$. Then we can divide through by c_n and move v_n to the other side of the equation to get

$$v_n = a_1 v_1 + \cdots + a_{n-1} v_{n-1}.$$

We claim that the list $(v_1, \dots, v_{n-1})$ is still spanning. For let v be any vector in V. By assumption, we can write v as a linear combination of $v_1, \dots, v_n$; say

$$\begin{aligned} v &= x_1v_1 + \cdots + x_{n-1}v_{n-1} + x_nv_n \\ &= x_1v_1 + \cdots + x_{n-1}v_{n-1} + x_n(a_1v_1 + \cdots + a_{n-1}v_{n-1}) \\ &= (x_1 + a_1x_n)v_1 + \cdots + (x_{n-1} + a_{n-1}x_n)v_{n-1}. \end{aligned}$$

Thus, we have expressed v in terms of $v_1, \dots, v_{n-1}$; so this list of vectors is spanning.

Now we continue throwing away vectors in this manner until we reach a sublist which is a basis.

To show that the given vectors are spanning, we have to show that we can express any vector as a linear combination of them: this can be done by solving some linear equations.

Furthermore, by solving equations, we find that

$$7\begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}1\\4\\5\end{bmatrix} - 2\begin{bmatrix}2\\3\\4\end{bmatrix} - 2\begin{bmatrix}1\\2\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix},$$

so we can drop the fourth vector from the list. Now a similar calculation shows that the first three really are linearly independent: the three equations resulting from

$$x\begin{bmatrix}1\\2\\3\end{bmatrix} + y\begin{bmatrix}1\\4\\5\end{bmatrix} + z\begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

have only the solution $x = y = z = 0$. Thus the first three vectors form a basis.