MTH6140 **Linear Algebra II**

Fields and vector spaces **Summary**

A *field* is an algebraic structure K in which we can add and multiply elements, such that the following laws hold:

Addition laws

(FA0) For any $a,b \in K$, there is a unique element $a+b \in K$.

(FA1) For all $a,b,c \in K$, we have $a+(b+c) = (a+b)+c$.

(FA2) There is an element $0 \in K$ such that $a+0 = 0+a = a$ for all $a \in K$.

(FA3) For any $a \in K$, there exists $-a \in K$ such that $a+(-a) = (-a)+a = 0$.

(FA4) For any $a,b \in K$, we have $a+b = b+a$.

Multiplication laws

(FM0) For any $a,b \in K$, there is a unique element $ab \in K$.

(FM1) For all $a,b,c \in K$, we have $a(bc) = (ab)c$.

(FM2) There is an element $1 \in K$ such that $a1 = 1a = a$ for all $a \in K$.

(FM3) For any $a \in K$ with $a \neq 0$, there exists $a^{-1} \in K$ such that $aa^{-1} = a^{-1}a = 1$.

(FM4) For any $a,b \in K$, we have $ab = ba$.

Distributive law

(D) For all $a,b,c \in K$, we have $a(b+c) = ab+ac$.

Note the similarity of the addition and multiplication laws. We say that $(K,+)$ is an *abelian group* if (FA0)–(FA4) hold. Then (FM0)–(FM4) say that $(K \setminus \{0\}, \cdot)$ is also an abelian group. (We have to leave out 0 because, as (FM3) says, 0 does not have a multiplicative inverse.)

Examples of fields include $\mathbb{Q}$ (the rational numbers), $\mathbb{R}$ (the real numbers), $\mathbb{C}$ (the complex numbers), and $\mathbb{F}_p$ (the integers mod p, for p a prime number).

Let K be a field. A *vector space* V over K is an algebraic structure in which we can add two elements of V, and multiply an element of V by an element of K (this is called *scalar multiplication*), such that the following rules hold:

Addition laws

(VA0) For any $u, v \in V$, there is a unique element $u + v \in V$.

(VA1) For all $u, v, w \in V$, we have $u + (v + w) = (u + v) + w$.

(VA2) There is an element $0 \in V$ such that $v + 0 = 0 + v = av$ for all $v \in V$.

(VA3) For any $v \in V$, there exists $-v \in V$ such that $v + (-v) = (-v) + v = 0$.

(VA4) For any $u, v \in K$, we have $u + v = v + u$.

Scalar multiplication laws

(VM0) For any $a \in K$, $v \in V$, there is a unique element $av \in V$.

(VM1) For any $a \in K$, $u, v \in V$, we have $a(u + v) = au + av$.

(VM2) For any $a, b \in K$, $v \in V$, we have $(a + b)v = av + bv$.

(VM3) For any $a, b \in K$, $v \in V$, we have $(ab)v = a(bv)$.

(VM4) For any $v \in V$, we have $1v = v$ (where 1 is the element given by (FM2)).

Again, we can summarise (VA0)–(VA4) by saying that $(V, +)$ is an abelian group.

The most important example of a vector space over a field K is the set K^n of all n-tuples of elements of K: the addition and scalar multiplication are defined by the rules

$$\begin{aligned}(u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n), \\ a(v_1, v_2, \ldots, v_n) &= (av_1, av_2, \ldots, av_n).\end{aligned}$$

You may assume this information, and are not expected to provide proofs of what is claimed here.