MTH6140 Linear Algebra II

Proof of the Exchange Lemma

Recall the statement of the Exchange Lemma:

> *Let V be a vector space over K. Suppose that the vectors $v_1, \ldots, v_n$ are linearly independent, and that the vectors $w_1, \ldots, w_m$ are linearly independent, where $m > n$. Then we can find a number i with $1 \leq i \leq m$ such that the vectors $v_1, \ldots, v_n, w_i$ are linearly independent.*

In the proof, we need the following result about solutions of systems of linear equations from Linear Algebra I; I will outline its proof.

> *Given a system* $(*)$
>
> $$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= 0, \\ &\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= 0 \end{aligned}$$
>
> *of homogeneous linear equations, where the number n of equations is strictly less than the number m of variables*[1]*, there exists a non-zero solution $(x_1, \ldots, x_m)$ (that is, $x_1, \ldots, x_m$ are not all zero).*

This can be proved by induction on the number of variables. If the coefficients $a_{11}, a_{21}, \ldots, a_{n1}$ are all zero, then putting $x_1 = 1$ and the other variables zero gives a solution. If one of these coefficients is non-zero, then we can use the corresponding equation to express x_1 in terms of the other variables, obtaining $n-1$ equations in $m-1$ variables. By the induction hypothesis, these new equations have a non-zero solution; computing the value of x_1 gives a solution to the original equations.

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[1] Such a system, where the number of equations is strictly less than the number of variables, is said to be *under-determined*.

Now we turn to the proof of the Exchange Lemma. Let us argue for a contradiction, by assuming that the result is false: that is, assume that none of the vectors w_i can be added to the list $(v_1,\dots,v_n)$ to produce a larger linearly independent list. This means that, for all j, the list $(v_1,\dots,v_n,w_i)$ is linearly dependent. So there are coefficients $c_1,\dots,c_n,d$, not all zero, such that

$$c_1v_1+\cdots+c_nv_n+dw_i=0.$$

We cannot have $d=0$; for this would mean that we had a linear combination of $v_1,\dots,v_n$ equal to zero, contrary to the hypothesis that these vectors are linearly independent. So we can divide the equation through by d, and take w_i to the other side, to obtain (changing notation slightly)

$$w_i=a_{1i}v_1+a_{2i}v_2+\cdots+a_{ni}v_n=\sum_{j=1}^{n}a_{ji}v_j.$$

We do this for each value of $i=1,\dots,m$.

Now take a non-zero solution to the set of equations $(*)$ above: that is,

$$\sum_{i=1}^{m}a_{ji}x_i=0$$

for $j=1,\dots,n$.

Multiplying the formula for w_i by x_i and adding, we obtain

$$x_1w_1+\cdots+x_mw_m=\sum_{j=1}^{n}\left(\sum_{i=1}^{m}a_{ji}x_i\right)v_j=0.$$

But the coefficients are not all zero, so this means that the vectors $(w_1,\dots,w_m)$ are not linearly dependent, contrary to hypothesis.

So the assumption that no w_i can be added to $(v_1,\dots,v_n)$ to get a linearly independent set must be wrong, and the proof is complete.