Queen Mary
University of London

B. Sc. Examination by course unit 2010

MTH6140 Linear Algebra II

Duration: 2 hours

Date and time: 4th June 2010, 1000h–1200h

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best 4 questions answered will be counted.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): O.M.Jenkinson

Question 1 Let A and B be matrices of the same size over a field K.

(a) [3 marks] What are the 3 types of *elementary row operation* which may be applied to A?

(b) [3 marks] What does it mean to say that A and B are *equivalent*?

(c) [4 marks] What is meant by the *canonical form for equivalence* for A, and how is the *rank* of A defined?

(d) [7 marks] Let

$$A = \begin{bmatrix} 6 & 3 \\ 1 & 6 \end{bmatrix}$$

be a matrix over the field $K = \mathbb{F}_{11}$ (the field of integers mod 11). What is the rank of A, and what is its canonical form for equivalence? Briefly justify your answer.

(e) [8 marks] How many 2×2 matrices A are there over $K = \mathbb{F}_2$ (the field of integers mod 2)?

List all of these matrices A whose rank is equal to 1.

List all of these matrices A whose rank is equal to 1 and whose square A^2 also has rank equal to 1.

Question 2 Let V and W be finite-dimensional vector spaces over a field K.

(a) [3 marks] What does it mean to say that a map $T : V \to W$ is *linear*?

(b) [4 marks] If $T : V \to W$ is linear, how are its *kernel* $\ker(T)$, its *nullity* $\mathrm{nul}(T)$, its *image* $\mathrm{Im}(T)$, and its *rank* $\mathrm{rank}(T)$ defined?

(c) [3 marks] State the Rank-Nullity Theorem.

(d) [5 marks] Prove that the map $S : \mathbb{C}^3 \to \mathbb{C}$ defined by

$$S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = x_1 + x_2 - 5x_3$$

is linear.

(e) [3 marks] Does there exist a linear map $T : \mathbb{R}^4 \to \mathbb{R}^2$ such that $\mathrm{rank}(T)^2 = \mathrm{nul}(T)$? Justify your answer.

(f) [3 marks] Does there exist a linear map $T : \mathbb{R}^6 \to \mathbb{R}^2$ such that $\mathrm{rank}(T)^2 = \mathrm{nul}(T)$? Justify your answer.

(g) [4 marks] Give an example of maps $S : \mathbb{R}^2 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ which are not themselves linear, but whose sum $S + T$ is linear.

Question 3 Let V be a finite-dimensional vector space over $\mathbb{R}$.

(a) [4 marks] What is meant by a *quadratic form* (on V)?

(b) [2 marks] What does it mean for two real symmetric matrices to be *congruent*?

(c) [3 marks] What does *Sylvester's Law of Inertia* assert about real symmetric matrices?

(d) [4 marks] Which real symmetric matrix A represents the real quadratic form

$$Q(x,y,z) = 4x^2 + 14y^2 + 5z^2 + 16xy - 8xz - 20yz \quad ?$$

(e) [7 marks] Find a diagonal matrix B which is congruent to the matrix A from part (d).

(f) [5 marks] Determine the set of real values α for which the symmetric real matrix $A = \begin{bmatrix} \alpha & \alpha^2 \\ \alpha^2 & \alpha^3 \end{bmatrix}$ is congruent to the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Question 4 Let V be a finite-dimensional vector space over a field K.

(a) [3 marks] If $v_1, \dots, v_n$ is a basis for V, how is the *dual basis* for its dual space V^* defined?

(b) [8 marks] Suppose that the vectors v_1, v_2, v_3 form a basis for $V = \mathbb{R}^3$, and that the dual basis for V^* is denoted by f_1, f_2, f_3.

Suppose a second basis w_1, w_2, w_3 for V is given by

$$w_1 = v_1 + 2v_2, \quad w_2 = 2v_2 - 4v_3, \quad w_3 = 8v_3\,.$$

If g_1, g_2, g_3 denotes the basis of V^* which is dual to w_1, w_2, w_3, then express each of g_1, g_2, g_3 as a linear combination of f_1, f_2, and f_3.

(c) [4 marks] For a linear map $T : V \to V$, how are its *characteristic polynomial* c_T and *minimal polynomial* m_T defined?

(d) [2 marks] What does the *Cayley–Hamilton theorem* assert about a linear map $T : V \to V$?

(e) [8 marks] Determine the characteristic and minimal polynomials of the real matrix

$$A = \begin{bmatrix} 0 & -4 & -6 \\ 1 & 4 & 3 \\ 0 & 0 & 2 \end{bmatrix},$$

Question 5 Let V be a vector space over a field K.

(a) [3 marks] What does it mean to say that a non-empty subset U of V is a *subspace* of V?

(b) [3 marks] If U_1 and U_2 are subspaces of V, how is $U_1 + U_2$ defined?

(c) [3 marks] What does it mean to say that the vectors $v_1, \ldots, v_n \in V$ are *spanning*?

(d) [4 marks] Is the set

$$U = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_3 \end{bmatrix} \in \mathbb{C}^4 : 2x_1 + 5x_2 - 3x_3 + 7x_4 = 2 \right\}$$

a subspace of $\mathbb{C}^4$? Prove your assertion.

(e) [4 marks] Give an example of a spanning set for $\mathbb{R}^2$ which is not a basis.

(f) [8 marks] Let i denote the square root of -1. Given bases $B = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$ and $B' = \left(\begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2-i \end{bmatrix} \right)$ for $\mathbb{C}^2$, write down the transition matrix $P_{B,B'}$ from B to B'.

Hence, or otherwise, determine the coordinate representation of $v = \begin{bmatrix} 2 \\ 4-3i \end{bmatrix}$ with respect to the basis B'.

Question 6 Let V be a finite-dimensional vector space over $\mathbb{R}$.

(a) [4 marks] What is meant by an *inner product* on V?

(b) [3 marks] Given an inner product on V, what does it mean to say that a basis $v_1, \ldots, v_n$ of V is *orthonormal*?

(c) [4 marks] If $T : V \to V$ is a linear map, what does it mean to say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of T? What does it mean to say that $v \in V$ is an *eigenvector* of T?

(d) [6 marks] Determine the eigenvalues of the map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x-y-z \\ -x+y+z \\ -x+y+z \end{bmatrix}.$$

(e) [8 marks] Suppose that $\mathbb{R}^3$ is equipped with the standard inner product. Find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of the map T from (d) above.

End of Paper