B.Sc. Examination by Course Unit

MTH6140 Linear Algebra II

Duration: 2 hours

Date and time: 4th June 2010

Solutions

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Question 1 (a) [3 marks]

The 3 types of elementary row operation are:

Type 1 Add a multiple of the jth row to the ith, where $j \neq i$.

Type 2 Multiply the ith row by a non-zero scalar.

Tyle 3 Interchange the ith and jth row, where $j \neq i$.

(b) [3 marks]

A and B are said to be *equivalent* if $B = PAQ$, where P and Q are invertible matrices (of sizes $m \times m$ and $n \times n$ respectively, if A and B are both $m \times n$).

(c) [4 marks]

The *canonical form for equivalence* for A is a matrix B which can be obtained from A via a succession of elementary row and column operations, and satisfies $B_{ii} = 1$ for $0 \leq i \leq r$, where $r \leq \min\{m,n\}$, and all other entries of B are zero.

The *rank* of A is the number r.

(d) [7 marks]

The first column in A is equal to twice the second column (since $12 \equiv 1$ modulo 11), so the (column) rank of A equals 1. The canonical form for equivalence is therefore $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(e) [2 marks]

There are $2^4 = 16$ such matrices.

[3 marks]

9 matrices have rank 1, namely $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, and $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

[3 marks]

Of the above, 6 have the property that their square is also of rank 1, namely:

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$.

Question 2 (a) [3 marks]

$T : V \to W$ is *linear* if

$$T(cv + c'v') = cT(v) + c'T(v')$$

for all $c, c' \in K$, $v, v' \in V$.

(Full marks for other equivalent definitions such as: $T(v+w) = T(v) + T(w)$ and $T(cv) = cT(v)$ for all $v, w \in V$, $c \in K$.)

(b) [4 marks]

$\ker(T) = \{v \in V : T(v) = 0\}$

$\mathrm{nul}(T) = \dim(\ker(T))$

$\mathrm{Im}(T) = T(V) = \{w \in W : w = T(v) \text{ for some } v \in V\}$

$\mathrm{rank}(T) = \dim(\mathrm{Im}(T))$.

(c) [3 marks]

The Rank-Nullity Theorem asserts that $\mathrm{rank}(T) + \mathrm{nul}(T) = \dim(V)$.

(d) [5 marks]

Given $v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{C}^3$ and $v' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} \in \mathbb{C}^3$, and $c, c' \in K = \mathbb{C}$, we have

$$S(cv + c'v') = cx_1 + c'x_1' + (cx_2 + c'x_2') - 5(cx_3 + c'x_3').$$

Rearranging gives

$$S(cv + c'v') = c(x_1 + x_2 - 5x_3) + c'(x_1' + x_2' - 5x_3') = cS(v) + c'S(v'),$$

so S is linear.

(e) [3 marks]

No, there does *not* exist such a linear map: by the Rank-Nullity Theorem, $(r, n) = (\mathrm{rank}(T), \mathrm{nul}(T)) = (0,4)$ or $(1,3)$ or $(2,2)$ or $(3,1)$ or $(4,0)$, and by inspection none of these pairs is such that $r^2 = n$.

(f) [3 marks]

Yes, such T do exist, for example T defined by $T(x_1, x_2, x_3, x_4, x_5, x_6) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

(g) [3 marks]

Many examples, e.g. let $S = -T$ be any non-zero constant map.

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Question 3 (a) [4 marks] $Q : V \to \mathbb{R}$ is a *quadratic form* if

(i) $Q(cv) = c^2 Q(v)$ for all $c \in \mathbb{R}$, $v \in V$, and

(ii) the function b defined by

$$b(v,w) = Q(v+w) - Q(v) - Q(w)$$

is a (obviously symmetric) bilinear form on V (i.e. it is a map $b : V^2 \to \mathbb{R}$ which is linear in both its arguments).

An alternative answer such as: "A quadratic form *in n variables* $x_1, \ldots, x_n$ *over is a polynomial*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$$

in the variables in which every term has degree two (that is, is a multiple of $x_i x_j$ *for some* i, j*), and each* A_{ij} *belongs to* $\mathbb{R}$*."*

would also score full marks.

1 mark would be deducted for more informal answers such as "A quadratic form is a function which, when written out in coordinates, is a polynomial in which every term has total degree 2 *in the variables."*

(b) [2 marks]

Two symmetric matrices A, B over the field $\mathbb{R}$ are *congruent* if $B = P^\top AP$ for some invertible real matrix P.

(c) [3 marks]

Sylvester's Law of Inertia asserts that any $n \times n$ real symmetric matrix A is congruent to a matrix of the form

$$\begin{bmatrix} I_s & O & O \\ O & -I_t & O \\ O & O & O \end{bmatrix}$$

for some (uniquely defined) s, t.

(d) [4 marks]

The symmetric matrix A representing Q is

$$\begin{bmatrix} 4 & 8 & -4 \\ 8 & 14 & -10 \\ -4 & -10 & 5 \end{bmatrix}.$$

(e) [7 marks]

The calculation

$$Q(x,y,z) = 4(x+2y-z)^2 - 2y^2 + z^2 - 4yz = 4(x+2y-z)^2 - 2(y+z)^2 + 3z^2$$

shows the diagonal matrix

$$B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

to be congruent to A (though any B with signature $+1$ is also correct).

[1 mark] for noting that the linear substitution P (i.e. $P = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$ in the above) giving the congruence $B = P^\top AP$ is indeed *invertible*.

(f) [5 marks]

The congruence holds for the set of all *strictly positive* real values of α (i.e. for all $\alpha > 0$).

Incorrect answers may score partial marks for including elements of the following working: Note that the matrix A represents the quadratic form

$$Q(x,y) = \alpha x^2 + 2\alpha^2 xy + \alpha^3 y^2 = \alpha(x+\alpha y)^2 = \alpha u^2 + 0.v^2 = 1.U^2 + 0.V^2\,,$$

where we set $u := x + \alpha y$, $v := y$, *and* $U := \sqrt{\alpha}u = \sqrt{\alpha}(x+\alpha y)$, $V := v = y$ *(we can take the square root of* α *if it is positive).*

So the (invertible) linear substitution $P_1 = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix}$ *gives*

$$P_1^\top AP_1 = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}$$

and the (invertible) linear substitution $P_2 = \begin{bmatrix} \sqrt{\alpha} & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \alpha^{-1/2} & 0 \\ 0 & 1 \end{bmatrix}$ *gives*

$$P_2^\top \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, setting $P = P_1P_2$ *gives that* $P^\top AP = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, *so the matrices A and* $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *are congruent.*

If $\alpha < 0$ *then A is congruent to* $\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$, *and if* $\alpha = 0$ *then A is equal, hence congruent, to* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. *So in these cases A is not congruent to* $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *(by Sylvester's law of inertia).*

Question 4 (a) [3 marks]

The basis $f_1,\dots,f_n$ dual to $v_1,\dots,v_n$ is defined by $f_i(v_i)=1$ for $1\le i\le n$ and $f_i(v_j)=0$ for $1\le i,j\le n$, $i\ne j$.

(b) [8 marks]

The transition matrix P from the vs to the ws is

$$P=\begin{bmatrix}1&0&0\\2&2&0\\0&-4&8\end{bmatrix},$$

so the transition matrix between the dual bases is

$$(P^{-1})^{\top}=\frac{1}{16}\begin{bmatrix}16&-16&-8\\0&8&4\\0&0&2\end{bmatrix}=\begin{bmatrix}1&-1&-1/2\\0&1/2&1/4\\0&0&1/8\end{bmatrix}.$$

The coordinates of the gs in the basis of fs are the columns of this matrix. In other words:

$$g_1=f_1\,,$$
$$g_2=-f_1\;+\;\frac{1}{2}f_2\,,$$
$$g_3=-\frac{1}{2}f_1\;+\frac{1}{4}\;f_2\;+\;\frac{1}{8}f_3\,.$$

(c) [4 marks]

The *characteristic polynomial* c_T is defined by $c_T(x)=\det(xI-T)$, in other words $c_T(x)=c_A(x)=\det(xI-A)$ for any matrix A which represents T (with respect to some basis of V).

The *minimal polynomial* m_T is the monic polynomial p of smallest possible degree such that $p(T)=0$.

(d) [2 marks]

The Cayley-Hamilton theorem asserts that $c_T(T)=0$.

(e) [8 marks]

The characteristic polynomial is

$$c_A(x)=\begin{vmatrix}x&4&6\\-1&x-4&-3\\0&0&x-2\end{vmatrix}=(x-2)^3\,.$$

The minimal polynomial is $m_A(x)=(x-2)^2$, because $(A-2I)^2=0$ (and $A-2I\ne 0$).

Question 5 (a) [3 marks]

A non-empty subset $U \subseteq V$ is a *subspace* of V if $cu + c'u' \in U$ for all $u, u' \in U$, $c, c' \in K$.

(b) [2 marks]

$U_1 + U_2 = \{u_1 + u_2 : u_1 \in U_1, u_2 \in U_2\}$.

(c) [3 marks]

$v_1, \ldots, v_n \in V$ are *spanning* if for every $v \in V$ there exist $c_1, \ldots, c_n \in K$ satisfying $v = c_1 v_1 + \ldots + c_n v_n$.

(d) [4 marks]

U is *not* a subspace of $\mathbb{C}^4$, since it does not contain the zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

(e) [4 marks]

One possible example is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

(f) [4 marks]

The transition matrix is

$$P_{B,B'} = \begin{bmatrix} i & 1 \\ 1 & 2-i \end{bmatrix}.$$

[4 marks]

The coordinate representation $[v]_{B'}$ can be computed as

$$[v]_{B'} = P_{B,B'}^{-1}[v]_B = \frac{1}{2i}\begin{bmatrix} 2-i & -1 \\ 1 & 2-i \end{bmatrix}\begin{bmatrix} 2 \\ 4-3i \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} i \\ 4i+1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 2-i/2 \end{bmatrix}.$$

(Alternatively, set $\begin{bmatrix} 2 \\ 4-3i \end{bmatrix} = a\begin{bmatrix} i \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2-i \end{bmatrix}$ and solve to find $a = 1/2$, $b = 2 - i/2$).

Question 6 (a) [4 marks]

An *inner product* on V is a function $b : V \times V \to \mathbb{R}$ satisfying

- b is bilinear (i.e. linear in the first variable when the second is kept constant and *vice versa*);
- b is *positive definite*, that is, $b(v,v) \geq 0$ for all $v \in V$, and $b(v,v) = 0$ if and only if $v = 0$.

(b) [3 marks]

The basis $v_1, \dots, v_n$ is called *orthonormal* if $b(v_i, v_i) = 1$ for $1 \leq i \leq n$ and $b(v_i, v_j) = 0$ for $1 \leq i, j \leq n$, $i \neq j$.

(c) [4 marks]

$\lambda \in K$ is an *eigenvalue* of T if there exists $v \in V \setminus \{0\}$ such that $T(v) = \lambda v$. Any such v is an *eigenvector* of T.

(d) [6 marks]

The eigenvalues are 0 and 3, since these are the roots of the characteristic polynomial

$$c_T(s) = \det \begin{bmatrix} s-1 & 1 & 1 \\ 1 & s-1 & -1 \\ 1 & -1 & s-1 \end{bmatrix} = (s-1)((s-1)^2-1) - 2s = s^2(s-3)\,.$$

(e) [8 marks: 4 for computing the eigenvectors, 4 for orthonormalising them]

One possible basis is

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

(Here the first vector is an eigenvector for 3. The second and third vectors are eigenvectors for 0, hence automatically orthogonal to the first vector, though care is needed in ensuring their mutual orthogonality).