Queen Mary
University of London

B. Sc. Examination by course unit 2009

MTH6140 Linear Algebra II

Duration: 2 hours

Date and time: 22 May 2009, 1000h–1200h

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best 4 questions answered will be counted.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): O.M.Jenkinson

Question 1 Let A be a matrix over a field K.

(a) [3 marks] What are the 3 types of *elementary column operation* which may be applied to A?

(b) [4 marks] What is meant by the *canonical form for equivalence* for A, and how is the *rank* of A defined?

(c) [10 marks] If

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 9 \end{bmatrix}$$

is a matrix over $\mathbb{R}$, find invertible matrices P and Q such that PAQ is in the canonical form for equivalence.

(d) [2 marks] What is the rank of the matrix A defined in (c) ?

(e) [2 marks] What does it mean to say that two square matrices (over a field K) are *similar*?

(f) [4 marks] Give an example, with justification, of two 2×2 matrices which have the same rank but are not similar.

Question 2 Let V and W be vector spaces over a field K.

(a) [3 marks] What does it mean to say that a map $T : V \to W$ is *linear*?

(b) [4 marks] If $T : V \to W$ is linear, how are its *kernel* $\ker(T)$, its *nullity* $\mathrm{nul}(T)$, its *image* $\mathrm{Im}(T)$, and its *rank* $\mathrm{rank}(T)$ defined?

(c) [5 marks] Prove that the map $S : \mathbb{R}^4 \to \mathbb{R}$ defined by

$$S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = 6x_2 - 3x_3 + x_4$$

is linear.

(d) [8 marks] For the map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1 + x_3 \\ x_2 \end{bmatrix},$$

find a basis for $\ker(T)$. Hence, or otherwise, compute $\mathrm{nul}(T)$ and $\mathrm{rank}(T)$.

(e) [5 marks] Does there exist a linear map $T : \mathbb{C}^5 \to \mathbb{C}^5$ such that $\mathrm{rank}(T) = \mathrm{nul}(T)$? Justify your assertion.

Question 3 Let K be a field whose characteristic is not equal to 2, and let V be a vector space over K.

(a) [5 marks] What is meant by a *bilinear form* (on V)? What does it mean to say that a bilinear form is *symmetric*?

(b) [4 marks] What is meant by a *quadratic form* (on V)?

(c) [3 marks] What does it mean for two symmetric matrices over K to be *congruent*?

(d) [4 marks] Which real symmetric matrix A represents the real quadratic form

$$Q(x,y,z) = 6x^2 + 4y^2 + z^2 + 12xy - 12xz - 4yz \quad ?$$

(e) [7 marks] Find a diagonal matrix B which is congruent to the matrix A from part (d).

(f) [2 marks] Compute the signature of the real symmetric matrix A from part (d).

Question 4 Let V be a finite-dimensional vector space over a field K.

(a) [4 marks] What is the definition of a *linear form* on V? How is the *dual space* V^* defined?

(b) [3 marks] If $v_1, \ldots, v_n$ is a basis for V, how is the *dual basis* for V^* defined?

(c) [4 marks] In the case when $K = \mathbb{R}$, what is meant by an *inner product* on V?

(d) [3 marks] Given an inner product on V as in (c), what does it mean to say that a basis $v_1, \ldots, v_n$ is *orthonormal*?

(e) [8 marks] Suppose that the vectors v_1, v_2, v_3 form a basis for $V = \mathbb{R}^3$, and that the dual basis for V^* is denoted by f_1, f_2, f_3.

Suppose a second basis w_1, w_2, w_3 for V is given by

$$w_1 = v_1 + 6v_2, \quad w_2 = 2v_2 + 4v_3, \quad w_3 = 2v_3\,.$$

If g_1, g_2, g_3 denotes the basis of V^* which is dual to w_1, w_2, w_3, then express each of g_1, g_2, g_3 as a linear combination of f_1, f_2, and f_3.

(f) [3 marks] Suppose that $V = \mathbb{R}^3$ is equipped with the standard inner product, and let U denote the subspace of $\mathbb{R}^3$ spanned by the vectors $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$. Give an orthonormal basis for U.

Question 5 Let V be a vector space over a field K.

(a) [3 marks] What does it mean to say that a non-empty subset U of V is a *subspace* of V?

(b) [5 marks] Is the set

$$U = \left\{ \begin{bmatrix} s+3t \\ t \\ 0 \end{bmatrix} : s,t \in \mathbb{R} \right\}$$

a subspace of $\mathbb{R}^3$? Prove your assertion.

(c) [3 marks] What does it mean to say that the vectors $v_1, \ldots, v_n \in V$ are *linearly independent*?

(d) [3 marks] What does it mean to say that the vectors $v_1, \ldots, v_n \in V$ are a *basis* for V?

(e) [3 marks] What does it mean to say that V is *finite-dimensional*? In this case how is its *dimension* $\dim(V)$ defined?

(f) [8 marks] Suppose that V is finite-dimensional, and that U is a subspace of V satisfying $\dim(U) = \dim(V)$. Prove that $U = V$.

Question 6 Let V be a finite-dimensional vector space over a field K, with $\dim(V) \geqslant 2$.

(a) [4 marks] If $T : V \to V$ is a linear map, what does it mean to say that $\lambda \in K$ is an *eigenvalue* of T? For an eigenvalue of T, how is the corresponding *eigenspace* defined?

(b) [4 marks] For a linear map $T : V \to V$, how are its *characteristic polynomial* c_T and *minimal polynomial* m_T defined?

(c) [3 marks] What does it mean to say that a linear map $P : V \to V$ is a *projection*?

(d) [6 marks] Show that if $P : V \to V$ is a projection, and $\mathrm{Im}(P) = V$, then P is the identity map on V (i.e. $P(v) = v$ for all $v \in V$).

(e) [6 marks] List, with justification, all those polynomials which arise as the minimal polynomial for some projection $P : V \to V$.

(f) [2 marks] List, with justification, those scalars $\lambda \in K$ which arise as the eigenvalue of some projection $P : V \to V$.

End of Paper