B. Sc. Examination by Course Unit 2009

MTH6140 Linear Algebra II

Duration: 2 hours

Date and time: ??

Solutions

Question 1 (a) [3 marks]

The 3 types of elementary column operation are:

Type 1 Add a multiple of the jth column to the ith, where $j \neq i$.

Type 2 Multiply the ith column by a non-zero scalar.

Tyle 3 Interchange the ith and jth column, where $j \neq i$.

(b) [4 marks]

The *canonical form for equivalence* for A is a matrix B which can be obtained from A via a succession of elementary row and column operations, and satisfies $B_{ii} = 1$ for $0 \leq i \leq r$, where $r \leq \min\{m,n\}$, and all other entries of B are zero.

The *rank* of A is the number r.

(c) [10 marks]

We can successively apply the following row and column operations:

Subtract twice the 1st column from the 2nd (i.e. right multiply by $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$).

Subtract 3 times the 1st column from the 3rd (i.e. right multiply by $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$).

Subtract 4 times the 1st row from the 2nd row (i.e. left multiply by $\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$).

Multiply the 2nd row by $-1/3$ (i.e. left multiply by $\begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix}$).

Subtract the 2nd column from the 3rd (i.e. right multiply by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$).

The resulting matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, the canonical form for equivalence for A, with

$$P = \begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4/3 & -1/3 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(However other valid row and column operations may lead to other valid P and Q, which will also score full marks).

(d) [2 marks]

A has rank 2, by inspection of its canonical form.

(e) [2 marks]

Two $n \times n$ matrices A, B are *similar* if there exists an invertible $n \times n$ matrix P (over the same field) such that $P^{-1}AP = B$.

(f) [4 marks]

The real matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ both have rank 2 but are not similar (their eigenvalues differ).

Question 2 (a) [3 marks]

$T : V \to W$ is *linear* if

$$T(cv + c'v') = cT(v) + c'T(v')$$

for all $c, c' \in K$, $v, v' \in V$.

(Full marks for other equivalent definitions such as: $T(v+w) = T(v) + T(w)$ and $T(cv) = cT(v)$ for all $v, w \in V$, $c \in K$.)

(b) [4 marks]

$\ker(T) = \{v \in V : T(v) = 0\}$

$\mathrm{nul}(T) = \dim(\ker(T))$

$\mathrm{Im}(T) = T(V) = \{w \in W : w = T(v) \text{ for some } v \in V\}$

$\mathrm{rank}(T) = \dim(\mathrm{Im}(T))$.

(c) [5 marks]

Given $v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4$ and $v' = \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \end{bmatrix} \in \mathbb{R}^4$, and $c, c' \in K = \mathbb{R}$, we have

$$S(cv + c'v') = 6(cx_2 + c'x'_2) - 3(cx_3 + c'x'_3) + (cx_4 + c'x'_4).$$

Rearranging gives

$$S(cv + c'v') = c(6x_2 - 3x_3 + x_4) + c'(6x'_2 - 3x'_3 + x'_4) = cS(v) + c'S(v'),$$

so S is linear.

(d) [5 marks] $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \ker(T)$ if and only if $x_2 = 0$ and $x_1 = -x_3$, so the single vector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is a basis for $\ker(T)$.

[3 marks] Consequently $\mathrm{nul}(T) = 1$ and $\mathrm{rank}(T) = 3 - \mathrm{nul}(T) = 2$.

(e) [5 marks]

There does *not* exist such a linear map, since otherwise

$$5 = \mathrm{rank}(T) + \mathrm{nul}(T) = 2\,\mathrm{rank}(T),$$

a contradiction.

Question 3 (a) [5 marks]

$b : V \times V \to K$ is a *bilinear form* if it is a linear function of each variable when the other is kept constant: that is, for all $v, w, v_1, v_2, w_1, w_2 \in V$, $c \in K$, we have:

$$b(v, w_1 + w_2) = b(v, w_1) + b(v, w_2), \qquad b(v, cw) = cb(v, w)$$

and

$$b(v_1 + v_2, w) = b(v_1, w) + b(v_2, w), \qquad b(cv, w) = cb(v, w)\,.$$

A bilinear form b is *symmetric* if $b(v, w) = b(w, v)$ for all $v, w \in V$.

(b) [4 marks]

$Q : V \to K$ is a *quadratic form* if

(i) $Q(cv) = c^2 Q(v)$ for all $c \in K$, $v \in V$, and

(ii) the function b defined by

$$b(v, w) = Q(v + w) - Q(v) - Q(w)$$

is a (obviously symmetric) bilinear form on V.

An alternative answer such as: "A quadratic form *in n variables* $x_1, \ldots, x_n$ *over a field K is a polynomial*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$$

in the variables in which every term has degree two (that is, is a multiple of $x_i x_j$ *for some i, j), and each* A_{ij} *belongs to K."*

also scores full marks, while 1 mark is deducted for more informal answers such as "A quadratic form is a function which, when written out in coordinates, is a polynomial in which every term has total degree 2 *in the variables."*

(c) [3 marks]

Two symmetric matrices A, A' over a field K are *congruent* if $A' = P^\top AP$ for some invertible matrix P.

(d) [4 marks]

The symmetric matrix A representing Q is

$$\begin{bmatrix} 6 & 6 & -6 \\ 6 & 4 & -2 \\ -6 & -2 & 1 \end{bmatrix}.$$

(e) [6 marks] The calculation

$$Q(x,y,z) = 6(x+y-z)^2 - 2y^2 - 5z^2 + 8yz = 6(x+y-z)^2 - 2(y-2z)^2 + 3z^2$$

shows the diagonal matrix

$$B = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

to be congruent to A (though any B with signature $+1$ is also correct).

[1 mark] for noting that the linear substitution P (i.e. $P = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ in the above) giving the congruence $B = P^\top AP$ is indeed *invertible*.

(f) [2 marks]

The signature of A equals the signature of B, which is $2-1 = +1$.

Question 4 (a) [4 marks]

A *linear form* on V is a linear map from V to K (where K is regarded as a 1-dimensional vector space over K).

The *dual space* V^* is the set of all linear forms on V.

(b) [3 marks]

The basis $f_1, \ldots, f_n$ dual to $v_1, \ldots, v_n$ is defined by $f_i(v_i) = 1$ for $1 \leq i \leq n$ and $f_i(v_j) = 0$ for $1 \leq i, j \leq n$, $i \neq j$.

(c) [4 marks]

An *inner product* on V is a function $b : V \times V \to \mathbb{R}$ satisfying

- b is bilinear (i.e. linear in the first variable when the second is kept constant and *vice versa*);
- b is *positive definite*, that is, $b(v,v) \geq 0$ for all $v \in V$, and $b(v,v) = 0$ if and only if $v = 0$.

(d) [3 marks]

The basis $v_1, \ldots, v_n$ is called *orthonormal* if $b(v_i, v_i) = 1$ for $1 \leq i \leq n$ and $b(v_i, v_j) = 0$ for $1 \leq i, j \leq n$, $i \neq j$.

(e) [8 marks]

The transition matrix P from the vs to the ws is

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix},$$

so the transition matrix between the dual bases is

$$(P^{-1})^\top = \frac{1}{4}\begin{bmatrix} 4 & -12 & 24 \\ 0 & 2 & -4 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 6 \\ 0 & 1/2 & -1 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

The coordinates of the gs in the basis of fs are the columns of this matrix. In other words:

$$g_1 = f_1\,,$$
$$g_2 = -3f_1 + \frac{1}{2}f_2\,,$$
$$g_3 = 6f_1 - f_2 + \frac{1}{2}f_3\,.$$

(f) [3 marks]

One possibility is the basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Question 5 (a) [3 marks]

A non-empty subset $U \subseteq V$ is a *subspace* of V if $cu + c'u' \in U$ for all $u, u' \in U$, $c, c' \in K$.

(b) [2 marks] U *is* a subspace of $\mathbb{R}^3$.

[3 marks] Proof: U is non-empty, and if $u = \begin{bmatrix} s+3t \\ t \\ 0 \end{bmatrix}, u' = \begin{bmatrix} s'+3t' \\ t' \\ 0 \end{bmatrix} \in U$ and $c, c' \in K = \mathbb{R}$, then

$$cu + c'u' = \begin{bmatrix} (cs + c's') + (3ct + 3c't') \\ ct + c't' \\ 0 \end{bmatrix} = \begin{bmatrix} S + 3T \\ T \\ 0 \end{bmatrix} \in U$$

where $S = cs + c's'$ and $T = ct + c't'$.

(c) [3 marks]

$v_1, \ldots, v_n \in V$ are *linearly independent* if whenever $c_1, \ldots, c_n \in K$ satisfy $c_1 v_1 + \ldots + c_n v_n = 0$ then necessarily $c_1 = \ldots = c_n = 0$.

(d) [3 marks]

$v_1, \ldots, v_n \in V$ are a *basis* for V if they are linearly independent but also *spanning*, in other words for every $v \in V$ there exist $c_1, \ldots, c_n \in K$ satisfying $v = c_1 v_1 + \ldots + c_n v_n$.

(e) [3 marks]

V is *finite-dimensional* if it has a (finite) basis. In this case its *dimension* is defined to be the number of elements in any basis (proved in the course to be independent of the basis chosen).

(f) [8 marks]

One possible proof is:

Let $\dim(U) = \dim(V) = n$, and choose a basis $u_1, \ldots, u_n$ for U. Then $u_1, \ldots, u_n$ are linearly independent vectors in V, a space of dimension n; so they form a basis for V. So in particular $u_1, \ldots, u_n$ is a spanning set; this means that every vector $v \in V$ is a linear combination of $u_1, \ldots, u_n$, and hence lies in U. So $V \subseteq U$. But we are given that $U \subseteq V$; so $U = V$.

Question 6 (a) [4 marks]

$\lambda \in K$ is an *eigenvalue* of T if there exists $v \in V \setminus \{0\}$ such that $T(v) = \lambda v$.

The corresponding *eigenspace* is the set $\{v \in V : T(v) = \lambda v\}$.

(b) [4 marks]

The *characteristic polynomial* c_T is defined by $c_T(x) = \det(xI - T)$, in other words $c_T(x) = c_A(x) = \det(xI - A)$ for any matrix A which represents T (with respect to some basis of V).

The *minimal polynomial* m_T is the monic polynomial p of smallest possible degree such that $p(T) = 0$.

(c) [3 marks]

It means that $P^2 = P$.

(d) [6 marks]

Let v be any vector in V. We have to show that $P(v) = v$. Now $\mathrm{Im}(P) = V$, so there exists $v' \in V$ such that $P(v') = v$. But P is a projection, so $P(v) = P(P(v')) = P(v') = v$, as required.

(e) [6 marks]

The possible polynomials are (i) x, (ii) $x - 1$, and (iii) $x(x-1)$.

To see that these are the only candidates note that $P^2 - P = 0$, so the minimal polynomial must divide $x(x-1)$.

Each of the 3 polynomials does indeed arise as the minimal polynomial of some projection, namely: (i) the zero map, (ii) the identity map, (iii) all other projections (e.g. $P\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix}$ on $V = K^2$).

(f) [2 marks]

0 and 1 are the only possible eigenvalues.

Proof: λ is an eigenvalue if and only if $x - \lambda$ divides $m_T(x)$. The result then follows from (f).

(Alternative proof: if λ is an eigenvalue, with eigenvector v, then $\lambda v = Pv = P^2 v = \lambda^2 v$, so $\lambda - \lambda^2 = 0$, hence $\lambda = 0$ or 1).