Non-periodic not everywhere dense trajectories in triangular billiards

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Abstract. Building on tools that have been successfully used in the study of rational billiards, such as induced maps and interval exchange transformations, we provide a construction of a one-parameter family of isosceles triangles exhibiting non-periodic trajectories that are not everywhere dense. In particular, this provides a definitive negative answer to a long-standing open question on the density of non-periodic trajectories in triangular billiards.

1. Introduction and results

Billiards, that is, the ballistic motion of a point particle in the plane with elastic collisions at the boundary, are among the simplest mechanical systems producing intricate dynamical features and thus serve as a paradigm in applied dynamical systems theory [**5**]. The seemingly trivial case of billiards with piecewise straight boundaries, known as polygonal billiards, offers surprisingly hard challenges [**8**]. When the inner angles of the polygon are rational multiples of π the billiard dynamics is dominated by a collection of conserved quantities and a rigorous and sophisticated machinery for its treatment becomes available, see, for example, [**7, 10**] for overviews. Little to nothing is known in the irrational case. A notable exception is the proof of ergodicity of the Lebesgue measure in a topologically large class of irrational polygonal billiards [**9**]. It is however still not clear whether such a topologically large class has implications for numerical simulations of the billiard dynamics. Numerical studies of polygonal billiards are inconclusive. Depending on the shape of the polygon, correlations in irrational billiards may or may not exhibit decay [**1, 4**], and even the ergodicity of the Lebesgue measure has been questioned [**12**]. Recently the relevance of symmetries has been emphasised as an explanation for this conundrum [**13**].

In this article we shall be concerned with the simplest examples of polygonal billiards, namely those of triangular shape. In particular we shall revisit a hypothesis formulated by Zemlyakov, see [**6**], according to which trajectories are either periodic or cover the billiard table densely. While [**6**] shows that this dichotomy does not hold in convex[1] polygonal billiards with more than three sides, the proof is flawed for triangular billiards, as pointed out recently in [**11**]. Thus the existence of non-periodic and not everywhere dense trajectories in triangular billiards remains an open problem. We will fill this gap by constructing trajectories of this

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[1] For a simpler case of non-convex billiards, McMullen constructed an L-shaped example for which this dichotomy fails.

type for a large set of symmetric triangular billiards. For this purpose, similarly to [**6**] we reduce this problem to the properties of an induced one-dimensional map, a technique more commonly used in the case of rational billiards. Leaving details of the notation for later sections, we will prove the following.

Theorem 1. *Consider a billiard map in the isosceles triangle with inner angles* $(\alpha, \alpha, \pi - 2\alpha)$ *and* $\alpha \in (\alpha_*, 3\pi/10)$ *for some* α_* *with* $\pi/4 < \alpha_* < 2\pi/7$. *Then there exist an angle* $\phi_* \in (0, \pi)$ *and an induced map on the base of the triangle* $\{[k = 1, \phi_*, x] : x \in [0, 1]\}$ *which is a rotation on the unit interval, that is,* $x \mapsto x + \omega \mod 1$ *with* $\omega = \cos(3\alpha)/(2\cos(\alpha)\cos(4\alpha))$.

Using the above result we are able to answer the hypothesis by Zemlyakov negatively.

Corollary 2. *For all values of* $\alpha \in (\alpha_*, 3\pi/10)$ *with* $\cos(5\alpha)/\cos(3\alpha) \in \mathbb{R} \setminus \mathbb{Q}$, *in particular for all algebraic* $\alpha \in (\alpha_*, 3\pi/10)$, *the billiard dynamics in the isosceles triangle contains trajectories which are non-periodic and not everywhere dense in the triangle.*

The main idea of the proof can be gleaned from the Zemlyakov-Katok unfolding of the billiard dynamics [**14**]. Unfolding the dynamics in a particular direction determined by a heteroclinic connection it can be seen that the dynamics takes place in two recurrent cylinders, see Figure 1. As a result it is possible to introduce an induced map (on the base of the triangle) which turns out to be an irrational rotation. The construction also reveals that the cylinders do not cover the whole interior of the triangle, thus yielding non-periodic and not everywhere dense trajectories, and together forming a non-trivial flat strip in the sense of [**3**].

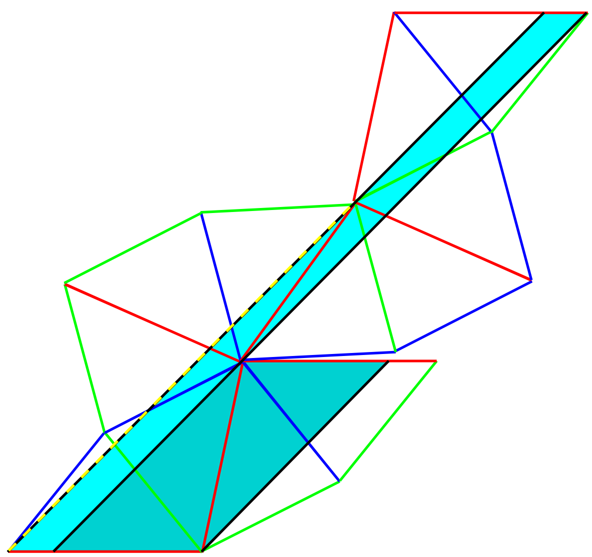

Figure 1. Diagrammatic view of the Zemlyakov-Katok unfolding of the billiard dynamics in an isosceles triangle with a 10- and a 4-recurrent cylinder, respectively (shaded). The heteroclinic connection is shown in dashed yellow. The three sides of the triangle have been coloured (base: red, right leg: green, left leg: blue).

We note that the proof of this corollary implies that the constructed billiard trajectories never visit a certain neighborhood around a tip of the isosceles triangle. This neighborhood can be replaced by a polygonal one, forming a convex n-gon for any $n \geq 4$, thus providing an alternative proof to Theorem 1 in [**6**] on the existence of non-periodic and not everywhere dense billard trajectories in any convex n-gon.

This article is organised as follows. After fixing notation in Section 2, the existence of the heteroclinic connection will be established by Lemma 8 in Section 3.2. We then turn to the existence and the properties of the two recurrent cylinders in Proposition 13 in Section 3.3 and Proposition 15 in Section 3.4, respectively. The symmetry of the triangle will be instrumental in setting up these cylinders and Lemma 12 of Section 3.3 summarises the main impact of the symmetry. The proof of the main results follows standard arguments and will be presented in Section 3.5.

While our particular construction works for a limited range of angles α, we suspect that the corollary holds for all isosceles triangles. Analogous constructions can be performed for other angles, but a more systematic approach would be needed to cover the general case. Even the condition on the symmetry can be relaxed as, for trivial reasons, an analogous statement holds in right-angled triangles. Above all our result rests on the existence of well-defined induced maps which deserves to be investigated in greater detail for irrational triangles.

2. Notation and billiard map

Consider a triangle with positively oriented boundary. The sides are labelled by a cyclic index $k = 1, 2, 3$. We denote by s_k the length of side k. The side with label $k = 1$ is called the base. We chose units of length such that $s_1 = 1$. Denote by γ_2 and γ_3 the left and right inner angle on the base, respectively. The angle opposite to the base is denoted by γ_1. We shall focus exclusively on the case of isosceles triangles, that is, $\gamma_2 = \gamma_3 = \alpha$. It readily follows that $s_2 = s_3 = 1/(2\cos(\alpha))$.

The ballistic motion of a point particle with elastic bounces on the sides of the triangle traces out a planar curve consisting of straight line segments. We call this curve the *trajectory*. We denote by $x_t^{[k]}$, $0 < x_t^{[k]} < s_k$, the location of the bounce of the particle (at discrete time t) at side k, and by $\phi_t^{[k]} \in (0, \pi)$ the angle between the oriented side and the outgoing ray of the trajectory. We call a move *counter-clockwise (ccw)* if a bounce on side k is followed by a bounce on side $k+1$. Similarly we call a move *clockwise (cw)* if a bounce on side k is followed by a bounce on side $k-1$. Subsequent bounces are related by the *billiard map*

$$(x_t^{[k_t]}, \phi_t^{[k_t]}) \mapsto (x_{t+1}^{[k_{t+1}]}, \phi_{t+1}^{[k_{t+1}]}) \tag{1}$$

where

$$\phi_{t+1}^{[k_{t+1}]} = \begin{cases} \pi - \phi_t^{[k_t]} - \gamma_{k_t-1} & \text{if} \quad k_{t+1} = k_t + 1 \text{ (ccw)} \\ \pi - \phi_t^{[k_t]} + \gamma_{k_t+1} & \text{if} \quad k_{t+1} = k_t - 1 \text{ (cw)} \end{cases} \tag{2}$$

$$x_{t+1}^{[k_{t+1}]} = \begin{cases} (s_{k_t} - x_t^{[k_t]}) \sin(\phi_t^{[k_t]}) / \sin(\phi_{t+1}^{[k_{t+1}]}) & \text{if} \quad k_{t+1} = k_t + 1 \text{ (ccw)} \\ s_{k_{t+1}} - x_t^{[k_t]} \sin(\phi_t^{[k_t]}) / \sin(\phi_{t+1}^{[k_{t+1}]}) & \text{if} \quad k_{t+1} = k_t - 1 \text{ (cw)} \end{cases} \tag{3}$$

As it will be useful to keep track of the sequence of bouncing sides, we use a sligthly non-standard notation and call an *orbit* a finite or infinite sequence of triplets $([k_t, \phi_t^{[k_t]}, x_t^{[k_t]}])_{t \in I}$ which obeys the billiard map (1). Each orbit corresponds to a trajectory in the plane, and vice versa. We call a point $[k, \phi^{[k]}, x^{[k]}]$ *singular*, if it corresponds to one of the corners of the triangle, that is, if $x^{[k]} = 0$ or $x^{[k]} = s_k$. We call an orbit *regular* if all its points are non-singular. An orbit which starts or ends at a singular point will be referred to as a *singular orbit*, while an orbit which starts and ends at a singular point is called a *heteroclinic connection*[2].

For a fixed side k of the triangle and a fixed angle ϕ, we will refer to the family of parallel trajectory segments reflecting from k at angle ϕ and returning to k with the same angle ϕ after a fixed sequence of bouncing sides as a *recurrent cylinder*.

3. Proof of results

Our proof consists of several steps. We will first establish the existence of a suitable induction angle, such that an orbit emanating from the left endpoint of the base at this angle forms a heteroclinic connection with a certain length-5 sequence of bouncing sides. We will then show that *every* orbit emanating from the base at this angle returns to the base with the same angle after a fixed number of bounces (either 10 or 4, depending on the initial point). The two corresponding sets of billiard trajectories will form two recurrent cylinders in the plane, crucially bounded away by a positive distance from one of the triangle's vertices. This construction will yield an induced map, forming an interval exchange transformation over two subintervals of the triangle base. The rotation number of this interval exchange transformation will depend continuously on the angle of the isosceles triangle, implying an irrational rotation and hence dense trajectories in the union of the recurrent cylinders for a large set of angles of the triangle.

3.1. Induction angle. We begin by proving several lemmas, which will be used to establish that for a suitable range of values of α there exists an angle ϕ_* (depending on α), such that the orbit emanating from the left endpoint of the base at angle ϕ_* forms a heteroclinic connection.

LEMMA 3. *For $\alpha \in (\pi/4, 3\pi/10)$ the equation $g(\alpha) = \sin(7\alpha) - \sin(3\alpha) + \sin(\alpha) = 0$ has a unique solution $\alpha_* \in (\pi/4, 2\pi/7)$.*

PROOF. We have that $g(\pi/4) = \sin(7\pi/4) < 0$ and $g(3\pi/10) = \sin(3\pi/10) > 0$. Since $7\alpha \in (7\pi/4, 21\pi/10)$, $3\alpha \in (3\pi/4, 9\pi/10)$ and $\alpha \in (\pi/4, 3\pi/10)$ it follows that

$$g'(\alpha) = 7\cos(7\alpha) - 3\cos(3\alpha) + \cos(\alpha) > 0\,.$$

Existence and uniqueness of $\alpha_* \in (\pi/4, 3\pi/10)$ now follow from a variant of the intermediate value theorem. For the remaining assertion observe that $g(2\pi/7) = \sin(2\pi/7) - \sin(6\pi/7) = \sin(2\pi/7) - \sin(\pi/7) > 0$. □

LEMMA 4. *Let $\alpha \in [\pi/4, 3\pi/10]$. The equation*

$$g(\alpha, \phi) = \sin(6\alpha + \phi) - \sin(2\alpha + \phi) + \sin(\phi) = 0 \tag{4}$$

has a unique solution $\phi = \phi_*(\alpha)$ in $(0, \pi)$.

PROOF. We have $g(\alpha, 0) = 2\sin(2\alpha)\cos(4\alpha) < 0$. Existence and uniqueness of the solution in $(0, \pi)$ follow from the observation that $g(\alpha, \phi)$ is a Fourier polynomial in ϕ containing only the two first order terms. □

LEMMA 5. *Let $\alpha \in (\alpha_*, 3\pi/10)$. The solution to (4) obeys*

$$0 < \phi_* < \alpha, \quad \alpha + \phi_* > \frac{\pi}{2}, \quad 3\alpha + \phi_* > \pi, \quad 6\alpha + \phi_* < 2\pi < 7\alpha + \phi_*\,. \tag{5}$$

[2]In the literature, it is often referred to as a *generalized diagonal*, which itself corresponds to a saddle connection in the unfolding.

Proof. Using the substitution

$$\alpha = 3\pi/10 + x, \quad \phi_* = \pi/5 + y$$

with $-\pi/20 < x < 0$ (equivalent to $\alpha \in (\pi/4, 3\pi/10)$), equation (4) reads

$$\bar{g}(x, y) = \sin(6x + y) + 2\sin(x + y)\sin(3\pi/10 + x) = 0\,. \tag{6}$$

We have that

$$\bar{g}(x, -x) = \sin(5x) < 0, \quad \bar{g}(x, -6x) = 2\sin(-5x)\sin(3\pi/10 + x) > 0\,.$$

It follows that $-x < y < -6x$ with $-\pi/20 < x < 0$, and therefore

$$\begin{aligned}
\alpha + \phi_* &= 3\pi/10 + \pi/5 + x + y > \pi/2,\\
6\alpha + \phi_* &= 9\pi/5 + \pi/5 + 6x + y < 2\pi,\\
3\alpha + \phi_* &= 9\pi/10 + 3x + \pi/5 + y = \pi + x + y + \pi/10 + 2x > \pi,\\
7\alpha + \phi_* &= 6\alpha + \alpha + \phi_* > 3\pi/2 + \pi/2\,.
\end{aligned}$$

Treating y in (6) as a function of x, implicit differentiation yields

$$\begin{aligned}
0 =& \frac{dy}{dx}\left(\cos(6x + y) + 2\cos(x + y)\sin(3\pi/10 + x)\right)\\
&+ 6\cos(6x + y) + 2\cos(x + y)\sin(3\pi/10 + x) + 2\sin(x + y)\cos(3\pi/10 + x)\,.
\end{aligned}$$

Since $-3\pi/10 < 6x + y < 0$, $0 < x + y < 3\pi/10$, and $\pi/4 < 3\pi/10 + x < 3\pi/10$, all trigonometric terms are positive and $dy/dx < 0$. Hence the solution $\phi_*(\alpha)$ is a strictly monotonic decreasing function for $\alpha \in (\pi/4, 3\pi/10)$. Since Lemma 3 and 4 imply $\phi_*(\alpha_*) = \alpha_*$ the final assertion follows. □

For the remainder of the paper we will refer to the value obtained in Lemma 3 as α_*, and for $\alpha \in (\alpha_*, 3\pi/10)$ we will write $\phi_* = \phi_*(\alpha)$, omitting the dependence on the angle α where there is no risk of ambiguity.

3.2. Heteroclinic orbit. Next, we proceed to show the existence of a heteroclinic orbit starting from the left endpoint of the base at angle ϕ_*. For this, we will ascertain that the formal recurrence equations (3) and (4) are satisfied by a given sequence of bouncing sides, angles, and spatial coordinates, which therefore form a valid (that is, realisable) orbit. We define the sequence of bouncing sides

$$(m_t)_{0 \le t \le 5} = (1, 2, 3, 1, 3, 1) \tag{7}$$

and introduce the sequence of angles

$$\begin{aligned}
&\psi_0 = \phi_*, && \psi_1 = \pi - \alpha - \phi_*, && \psi_2 = -\pi + 3\alpha + \phi_*,\\
&\psi_3 = 2\pi - 4\alpha - \phi_*, && \psi_4 = -\pi + 5\alpha + \phi_*, && \psi_5 = 2\pi - 6\alpha - \phi_*\,.
\end{aligned} \tag{8}$$

It is straightforward to check that the angles (8) together with (7) satisfy the recurrence (2). Furthermore, Lemma 5 yields the following result.

Lemma 6. *Let $\alpha \in (\alpha_*, 3\pi/10)$. The angles defined by (7) and (8) obey $0 < \psi_t < \pi$, $0 \le t \le 5$.*

Define, for $\delta \in \mathbb{R}$, the spatial coordinates

$$\begin{aligned}
\xi_0(\delta) &= \delta, \\
\xi_1(\delta) &= (s_1 - \delta)\frac{\sin(\psi_0)}{\sin(\psi_1)}, \\
\xi_2(\delta) &= s_2\frac{\sin(\psi_1)}{\sin(\psi_2)} - (s_1 - \delta)\frac{\sin(\psi_0)}{\sin(\psi_2)}, \\
\xi_3(\delta) &= s_3\frac{\sin(\psi_2)}{\sin(\psi_3)} - s_2\frac{\sin(\psi_1)}{\sin(\psi_3)} + (s_1 - \delta)\frac{\sin(\psi_0)}{\sin(\psi_3)}, \\
\xi_4(\delta) &= s_3 - s_3\frac{\sin(\psi_2)}{\sin(\psi_4)} + s_2\frac{\sin(\psi_1)}{\sin(\psi_4)} - (s_1 - \delta)\frac{\sin(\psi_0)}{\sin(\psi_4)}, \\
\xi_5(\delta) &= s_3\frac{\sin(\psi_2)}{\sin(\psi_5)} - s_2\frac{\sin(\psi_1)}{\sin(\psi_5)} + (s_1 - \delta)\frac{\sin(\psi_0)}{\sin(\psi_5)}\,.
\end{aligned} \tag{9}$$

It is again straightforward to check that the expressions in (9) together with (7) and (8) obey the formal recurrence in (3).

LEMMA 7. *Let $\alpha \in (\alpha_*, 3\pi/10)$. The coordinates defined in (7), (8), and (9) obey $\xi_0(0) = 0$, $\xi_5(0) = 1$, and $0 < \xi_t(0) < s_{k_t}$, $1 \le t \le 4$.*

PROOF. The initial coordinate $\xi_0(0) = 0$ is obvious. From Lemma 4 we have

$$\begin{aligned}
0 &= 2\cos(\alpha)\left(\sin(6\alpha + \phi_*) - \sin(2\alpha + \phi_*) + \sin(\phi_*)\right) \\
&= 2\cos(\alpha)\sin(6\alpha + \phi_*) - \sin(3\alpha + \phi_*) - \sin(\alpha + \phi_*) + 2\cos(\alpha)\sin(\phi_*)\,.
\end{aligned}$$

With (8) and $1 = s_1 = 2\cos(\alpha)s_{2/3}$ this reads

$$\sin(\psi_5) = s_3\sin(\psi_2) - s_2\sin(\psi_1) + s_1\sin(\psi_0)\,, \tag{10}$$

which implies $\xi_5(0) = 1$.

Using (10) and Lemma 6 we have

$$\xi_4(0) = s_3 - \sin(\psi_5)/\sin(\psi_4) < s_3\,.$$

Furthermore, by Lemma 5 we have

$$\sin(\psi_4) - 2\cos(\alpha)\sin(\psi_5) = \sin(7\alpha + \phi_*) > 0\,,$$

which implies $\xi_4(0) > 0$.

Again using (10) and Lemma 6 we have

$$\xi_3(0) = \sin(\psi_5)/\sin(\psi_3) > 0\,,$$

and by Lemma 5 and $2\alpha > \pi/2$ we obtain

$$\sin(\psi_3) - \sin(\psi_5) = 2\sin(\alpha)\cos(5\alpha + \phi_*) > 0\,,$$

which implies $\xi_3(0) < 1 = s_1$.

Lemma 5 implies

$$0 < \sin(\alpha - \phi_*) = \sin(\alpha + \phi_*) - 2\cos(\alpha)\sin(\phi_*)$$

so that, using the abbreviations (8) we have

$$\sin(\psi_0) < s_2\sin(\psi_1)\,. \tag{11}$$

Hence $\xi_2(0) > 0$. Furthermore, (10) and Lemma 6 yield

$$0 < s_3 - s_2 \frac{\sin(\psi_1)}{\sin(\psi_2)} + s_1 \frac{\sin(\psi_0)}{\sin(\psi_2)} \,,$$

which is equivalent to $\xi_2(0) < s_3$.

Finally, $\xi_1(0) > 0$ is obvious, and $\xi_1(0) < s_2$ follows from (11). □

Lemma 6 and 7 now yield the following conclusion.

Lemma 8. *Let $\alpha \in (\alpha_*, 3\pi/10)$. With the definitions (7), (8), and (9) the sequence $([m_t, \psi_t, \xi_t(0)])_{0\le t\le 5}$ defines a heteroclinic connection.*

3.3. Recurrent cylinder of length ten. In this section we will establish the existence of a point x_D on the base of the triangle, such that all points in $(0, x_D) \times \{\phi_*\}$ share the same length-10 sequence of bouncing sides. Using a symmetry of the triangle, this sequence will be shown to consist of the length-5 sequence (7), followed by a 'mirrored' variant of the same sequence, in a sense made precise below. Moreover we will observe that the image of $(0, x_D) \times \{\phi_*\}$ under the 10th iteration of the billiard map is $(1 - x_D, 1) \times \{\phi_*\}$. The orbit of the point (x_D, ϕ_*) itself will be singular, giving rise to a discontinuity of the induced map on the base. We begin by defining

$$x_D = 1 - \frac{\sin(2\alpha + \phi_*)}{\sin(\phi_*)} \,. \tag{12}$$

Lemma 9. *Let $\alpha \in (\alpha_*, 3\pi/10)$. The quantity defined by (12) obeys $x_D \in (0, 1)$ and*

$$x_D = \frac{\sin(\psi_5)}{\sin(\psi_0)} = 1 - \frac{\cos(3\alpha)}{2\cos(\alpha)\cos(4\alpha)} \,.$$

Proof. Lemma 5 implies $2\alpha + \phi_* < \pi$ so that $x_D < 1$. Furthermore

$$\sin(2\alpha + \phi_*) - \sin(\phi_*) = 2\sin(\alpha)\cos(\alpha + \phi_*) < 0$$

so that $x_D > 0$. Using Lemma 4 we have

$$x_D = \frac{-\sin(6\alpha + \phi_*)}{\sin(\phi_*)} = \frac{\sin(\psi_5)}{\sin(\psi_0)} \,.$$

Furthermore, (4) yields

$$(\cos(6\alpha) - \cos(2\alpha) + 1)\sin(\phi_*) + (\sin(6\alpha) - \sin(2\alpha))\cos(\phi_*) = 0$$

so that

$$\begin{aligned} \frac{\sin(2\alpha + \phi_*)}{\sin(\phi_*)} &= -\sin(2\alpha)\frac{\cos(6\alpha) - \cos(2\alpha) + 1}{\sin(6\alpha) - \sin(2\alpha)} + \cos(2\alpha) \\ &= \frac{\sin(4\alpha) - \sin(2\alpha)}{\sin(6\alpha) - \sin(2\alpha)} = \frac{\cos(3\alpha)}{2\cos(\alpha)\cos(4\alpha)} \,. \end{aligned}$$

□

Lemma 10. *Let $\alpha \in (\alpha_*, 3\pi/10)$. The coordinates defined in (7), (8), and (9) obey $0 < \xi_t(x_D) < s_{m_t}$, $t = 0, 1$, $\xi_2(x_D) = \xi_4(x_D) = s_3$, and $\xi_3(x_D) = \xi_5(x_D) = 0$.*

Proof. Since $\xi_0(x_D) = x_D$, Lemma 9 yields the assertion for $t = 0$.

Using (10) and Lemma 9 we have

$$\xi_5(x_D) = 1 - x_D \frac{\sin(\psi_0)}{\sin(\psi_5)} = 0 .$$

The assertions $\xi_3(x_D) = 0$ and $\xi_2(x_D) = \xi_4(x_D) = s_3$, follow from the equalities $\xi_3(\delta) = \xi_5(\delta)\sin(\psi_5)/\sin(\psi_3)$, $\xi_2(\delta) = s_3 - \xi_5(\delta)\sin(\psi_5)/\sin(\psi_2)$, and $\xi_4(\delta) = s_3 - \xi_5(\delta)\sin(\psi_5)/\sin(\psi_4)$. By Lemma 9 we have $\sin(\psi_0) > \sin(\psi_5)$, which implies

$$\xi_1(x_D) = \frac{\sin(\psi_0)}{\sin(\psi_1)} - \frac{\sin(\psi_5)}{\sin(\psi_1)} > 0 .$$

Finally, using (11) we obtain

$$\xi_1(x_D) = \frac{\sin(\psi_0)}{\sin(\psi_1)} - \frac{\sin(\psi_5)}{\sin(\psi_1)} < \frac{\sin(\psi_0)}{\sin(\psi_1)} < s_2 .$$

□

Since the angles and spatial coordinates defined in (7), (8), and (9) obey the formal recurrence scheme determined by the billiard map (1), Lemmas 6 and 10 yield the following result.

Lemma 11. *Let $\alpha \in (\alpha_*, 3\pi/10)$. Then, for any $\delta \in (0, x_D)$, the sequence $([m_t, \psi_t, \xi_t(\delta)])_{0\le t\le 5}$ with components defined by (7), (8), and (9) constitutes a regular orbit of the billiard map (1).*

The symmetry of the triangle has implications for the structure of orbits. Reflecting an orbit at the symmetry axis of the triangle yields again an orbit. In formal terms, this type of reflection is expressed as $[k, \phi^{[k]}, x^{[k]}] \mapsto [\bar{k}, \pi - \phi^{[k]}, s_k - x^{[k]}]$ where the adjoint index $\bar{k}$ is given by $\bar{1} = 1$, $\bar{2} = 3$, $\bar{3} = 2$. Similarly, reversing the motion gives again an orbit. In formal terms, the corresponding transformation reads $[k, \phi^{[k]}, x^{[k]}] \mapsto [k, \pi - \phi^{[k]}, x^{[k]}]$. Combining both operations maps an orbit to another orbit.

Lemma 12. *If $([k_t, \phi_t^{[k_t]}, x_t^{[k_t]}])_{0\le t\le T}$ denotes a finite regular orbit in a symmetric triangular billiard then $([\ell_t, \varphi_t^{[\ell_t]}, z_t^{[\ell_t]}])_{0\le t\le T}$ gives a finite regular orbit of the same length where $\ell_t = \bar{k}_{T-t}$, $\varphi_t^{[\ell_t]} = \phi_{T-t}^{[k_{T-t}]}$ and $z_t^{[\ell_t]} = s_{k_{T-t}} - x_{T-t}^{[k_{T-t}]}$. Here $\bar{k}$ denotes the adjoint index defined by $\bar{1} = 1$, $\bar{2} = 3$, $\bar{3} = 2$.*

Proof. We first note the identity $\overline{k \pm 1} = \bar{k} \mp 1$. The symmetry of the triangle is equivalent to $\gamma_k = \gamma_{\bar{k}}$ and $s_k = s_{\bar{k}}$. We consider a fixed time t, $0 \le t < T$.

Case A: Assume that the move $T-t-1 \to T-t$ in the original orbit is counter-clockwise, that is, $k_{T-t} = k_{T-t-1}+1$. Then $\ell_t = \bar{k}_{T-t} = \bar{k}_{T-t-1} - 1 = \ell_{t+1} - 1$ (that is, the move $t \to t+1$ in the image orbit is counter-clockwise as well).

Equation (2) tells us that for the original angles we have

$$\phi_{T-t}^{[k_{T-t}]} = \pi - \phi_{T-t-1}^{[k_{T-t-1}]} - \gamma_{k_{T-t-1}-1} .$$

Observing that

$$\gamma_{k_{T-t-1}-1} = \gamma_{\overline{k_{T-t-1}-1}} = \gamma_{\bar{k}_{T-t-1}+1} = \gamma_{\ell_{t+1}+1} = \gamma_{\ell_t - 1} ,$$

we have

$$\varphi_t^{[\ell_t]} = \pi - \varphi_{t+1}^{[\ell_{t+1}]} - \gamma_{\ell_t - 1}$$

which is the angle dynamics of the billiard map for the image orbit.

Similarly, (3) implies for the spatial coordinates of the original orbit that

$$x_{T-t}^{[k_{T-t}]} = \left(s_{k_{T-t-1}} - x_{T-t-1}^{[k_{T-t-1}]}\right) \frac{\sin(\phi_{T-t-1}^{[k_{T-t-1}]})}{\sin(\phi_{T-t}^{[k_{T-t}]})}$$

so that

$$s_{k_{T-t}} - z_t^{[\ell_t]} = z_{t+1}^{[\ell_{t+1}]} \frac{\sin(\varphi_{t+1}^{[\ell_{t+1}]})}{\sin(\varphi_t^{[\ell_t]})} .$$

Recalling that $s_{k_{T-t}} = s_{\bar{k}_{T-t}} = s_{\ell_t}$ we obtain the position dynamics of the billiard map for the image orbit.

Case B: The proof in case the move $T-t-1 \to T-t$ in the original orbit is clockwise, that is, $k_{T-t} = k_{T-t-1} - 1$, is similar. □

The symmetry allows us to extend the regular orbit derived in Lemma 11 to a recurrent orbit with $\phi_0^{[k_0]} = \phi_T^{[k_T]}$.

Proposition 13. *Let $\alpha \in (\alpha_*, 3\pi/10)$. For any $\delta \in (0, x_D)$ there exists a recurrent regular orbit of length 10 given by $([k_t, \phi_t^{[k_t]}, x_t^{[k_t]}])_{0\le t\le 10}$ with initial condition $[k_0, \phi_0^{[k_0]}, x_0^{[k_0]}] = [1, \phi_*, \delta]$ and endpoint $[k_{10}, \phi_{10}^{[k_{10}]}, x_{10}^{[k_{10}]}] = [1, \phi_*, \delta + 1 - x_D]$. The explicit expression for the orbit is given by $k_t = m_t$, $\phi_t^{[k_t]} = \psi_t$, and $x_t^{[k_t]} = \xi_t(\delta)$ for $0 \le t \le 5$ and $k_t = \bar{m}_{10-t}$, $\phi_t^{[k_t]} = \psi_{10-t}$, and $x_t^{[k_t]} = s_{\bar{m}_{10-t}} - \xi_{10-t}(x_D - \delta)$ for $6 \le t \le 10$.*

Proof. Let $\delta \in (0, x_D)$. Lemma 11 provides us with the regular orbit of length 5, $([m_t, \psi_t, \xi_t(\delta)])_{0\le t\le 5}$, with initial condition $[1, \phi_*, \delta]$ and endpoint $[1, \psi_5, \xi_5(\delta)]$. Replacing δ by $x_D - \delta$, Lemma 11 yields the regular length-5 orbit given by $([m_t, \psi_t, \xi_t(x_D - \delta)])_{0\le t\le 5}$. Applying Lemma 12 we obtain the regular orbit $([\bar{m}_{5-t}, \psi_{5-t}, s_{\bar{m}_{5-t}} - \xi_{5-t}(x_D - \delta)])_{0\le t\le 5}$ with initial condition $[1, \psi_5, s_1 - \xi_5(x_D - \delta)]$ and endpoint $[1, \phi_*, 1 - x_D + \delta]$. Recalling that (9) and (10) imply $\xi_5(\delta) = 1 - \delta \sin(\psi_0)/\sin(\psi_5)$ and using Lemma 9, we obtain $s_1 - \xi_5(x_D - \delta) = \xi_5(\delta)$. The assertions of the proposition follow by transitivity of orbits of the billiard map. □

3.4. Recurrent cylinder of length four. In an analogous way we can define a recurrent cylinder of length 4 for initial conditions $x_0^{[1]}$ in $(x_D, 1)$. For that purpose we define

$$(\ell_t)_{0\le t\le 2} = (1, 2, 1), \tag{13}$$

$$\theta_0 = \phi_*, \quad \theta_1 = \pi - \alpha - \phi_*, \quad \theta_2 = 2\alpha + \phi_*, \tag{14}$$

$$\begin{aligned} \eta_0(\delta) &= \delta, \\ \eta_1(\delta) &= (s_1 - \delta)\frac{\sin(\theta_0)}{\sin(\theta_1)}, \\ \eta_2(\delta) &= s_1 - (s_1 - \delta)\frac{\sin(\theta_0)}{\sin(\theta_2)} . \end{aligned} \tag{15}$$

Formally, the angles and spatial coordinates defined in (13), (14), and (15) obey the recursion scheme of the billiard map (1). In a similar vein to Lemma 11 we have the following result.

Lemma 14. *Let $\alpha \in (\alpha_*, 3\pi/10)$. Then, for any $\delta \in (x_D, 1)$, the sequence $([\ell_t, \theta_t, \eta_t(\delta)])_{0\le t\le 2}$ with components defined by (13), (14), and (15) constitutes a regular orbit of the billiard map (1).*

Proof. By Lemma 5 we have that $0 < \theta_t < \pi$ for $0 \le t \le 2$. Fixing $\delta \in (x_D, 1)$, Lemma 9 yields $0 < \eta_0(\delta) < 1$. We next observe that $\eta_2(1) = 1$, while (12) yields $\eta_2(x_D) = 0$, and hence $0 < \eta_2(\delta) < 1$.

Finally, we have $\eta_1(1) = 0$; furthermore, since

$$2\cos(\alpha)\sin(2\alpha + \phi_*) - \sin(\alpha + \phi_*) = \sin(3\alpha + \phi_*) < 0$$

by Lemma 5, we have

$$\eta_1(x_D) = \frac{\sin(2\alpha + \phi_*)}{\sin(\alpha + \phi_*)} < \frac{1}{2\cos(\alpha)},$$

and so $0 < \eta_1(x_D) < s_2$, which yields $0 < \eta_1(\delta) < s_2$ for $\delta \in (x_D, 1)$. □

Again employing the symmetry of the triangle yields the following result.

Proposition 15. *Let $\alpha \in (\alpha_*, 3\pi/10)$. For any $\delta \in (x_D, 1)$ there exists a recurrent regular orbit of length 4 given by $([k_t, \phi_t^{[k_t]}, x_t^{[k_t]}])_{0\le t\le 4}$ with initial condition $[k_0, \phi_0^{[k_0]}, x_0^{[k_0]}] = [1, \phi_*, \delta]$ and endpoint $[k_4, \phi_4^{[k_4]}, x_4^{[k_4]}] = [1, \phi_*, \delta - x_D]$. The explicit expression for the orbit is given by $k_t = \ell_t$, $\phi_t^{[k_t]} = \theta_t$, and $x_t^{[k_t]} = \eta_t(\delta)$ for $0 \le t \le 2$ and $k_t = \bar{\ell}_{4-t}$, $\phi_t^{[k_t]} = \theta_{4-t}$, and $x_t^{[k_t]} = s_{\bar{\ell}_{4-t}} - \eta_{4-t}(1 + x_D - \delta)$ for $3 \le t \le 4$.*

Proof. Let $\delta \in (x_D, 1)$. Lemma 14 provides us with the regular orbit of length 2, $([\ell_t, \theta_t, \eta_t(\delta)])_{0\le t\le 2}$, with initial condition $[1, \phi_*, \delta]$ and endpoint $[1, \theta_2, \eta_2(\delta)]$. Replacing δ by $1 + x_D - \delta$, Lemma 14 yields the regular length-2 orbit given by $([\ell_t, \theta_t, \eta_t(1 + x_D - \delta)])_{0\le t\le 2}$. Applying Lemma 12 we obtain the regular orbit $([\bar{\ell}_{s-t}, \theta_{s-t}, s_{\bar{\ell}_{2-t}} - \eta_{2-t}(1 + x_D - \delta)])_{0\le t\le 2}$ with initial condition $[1, \theta_2, s_1 - \eta_2(1 + x_D - \delta)]$ and endpoint $[1, \phi_*, \delta - x_D]$. Recalling that (12), (14) and (15) imply $x_D = 1 - \sin(\theta_2)/\sin(\theta_0)$ and $\eta_2(\delta) = 1 - (1 - \delta)\sin(\theta_0)/\sin(\theta_2)$ we obtain $s_1 - \eta_2(1 + x_D - \delta) = \eta_2(\delta)$. The assertions of the proposition follow by transitivity of orbits of the billiard map. □

3.5. Proof of the theorem and its corollary. Propositions 13 and 15 constitute the proof of Theorem 1 with the expression for the rotation number $\omega = 1 - x_D$ following readily from Lemma 9. The proof of the corollary will be based on the following lemma which summarises the findings in Lemma 7 and 14.

Lemma 16. *Let $\alpha \in (\alpha_*, 3\pi/10)$. There exists $\varepsilon > 0$ such that any infinite regular orbit $([k_t, \phi_t^{[k_t]}, x_t^{[k_t]})_{t\ge 0}$ of the billiard map with initial condition $([k_0, \phi_0^{[k_0]}, x_0^{[k_0]}] = [1, \phi_*, x_0^{[1]}]$ satisfies the conditions $x_t^{[k_t]} \le s_2 - \varepsilon$ whenever $k_t = 2$, and $x_t^{[k_t]} \ge \varepsilon$ whenever $k_t = 3$.*

Proof. The spatial coordinate $x_t^{[k_t]}$ does not take the values 0,1, or x_D if $k_t = 1$ as those are singularities or are mapped to singularities, see Lemma 10. Furthermore, by Proposition 13 and 15 the orbit is recurrent. Hence it is sufficient to consider the 4- and 10-recurrent pieces of the orbit.

Consider $[k_0, \phi_0^{[k_0]}, x_0^{[k_0]}] = [1, \phi_*, \delta]$ with $x_D < \delta < 1$, that is, a piece of the orbit in a 4-recurrent cylinder. Since by (12) and (15)

$$\eta_1(\delta) \le \eta_1(x_D) = \frac{\sin(\theta_2)}{\sin(\theta_1)}$$

$$s_3 - \eta_1(1 + x_D - \delta) \ge s_3 - \eta_1(x_D) = s_3 - \frac{\sin(\theta_2)}{\sin(\theta_1)}$$

we conclude from Proposition 15 that for $0 \le t \le 4$ we have $x_t^{[k_t]} \le \sin(\theta_2)/\sin(\theta_1)$ whenever $k_t = 2$ and $x_t^{[k_t]} \ge s_3 - \sin(\theta_2)/\sin(\theta_1)$ whenever $k_t = 3$.

Similarly, consider $[k_0, \phi_0^{[k_0]}, x_0^{[k_0]}] = [1, \phi_*, \delta]$ with $0 < \delta < x_D$, that is, a part of the orbit in a 10-recurrent cylinder. Then by (9) and (10)

$$\begin{aligned}
\xi_1(\delta) \le \xi_1(0) &= \frac{\sin(\psi_0)}{\sin(\psi_1)} \\
\xi_2(\delta) \ge \xi_2(0) &= s_3 - \frac{\sin(\psi_5)}{\sin(\psi_2)} \\
\xi_4(\delta) \ge \xi_4(0) &= s_3 - \frac{\sin(\psi_5)}{\sin(\psi_4)} \\
s_2 - \xi_4(x_D - \delta) \le s_2 - \xi_4(0) &= \frac{\sin(\psi_5)}{\sin(\psi_4)} \\
s_2 - \xi_2(x_D - \delta) \le s_2 - \xi_2(0) &= \frac{\sin(\psi_5)}{\sin(\psi_2)} \\
s_3 - \xi_1(x_D - \delta) \ge s_3 - \xi_1(0) &= s_3 - \frac{\sin(\psi_0)}{\sin(\psi_1)} \,.
\end{aligned}$$

Hence we conclude from Proposition 13 that for $0 \le t \le 10$ we have $x_t^{[k_t]} \le \varepsilon$ whenever $k_t = 2$ and $x_t^{[k_t]} \ge s_3 - \varepsilon$ whenever $k_t = 3$, where $\varepsilon = \min\{s_2 - \sin(\psi_0)/\sin(\psi_1), s_2 - \sin(\psi_5)/\sin(\psi_2), s_2 - \sin(\psi_5/\sin(\psi_4)\}$.

Altogether, the claim of the lemma is valid with the choice

$$\varepsilon = \min\left\{ s_2 - \frac{\sin(\theta_2)}{\sin(\theta_1)}, s_2 - \frac{\sin(\psi_0)}{\sin(\psi_1)}, s_2 - \frac{\sin(\psi_5)}{\sin(\psi_2)}, s_2 - \frac{\sin(\psi_5)}{\sin(\psi_4)} \right\} .$$

Lemma 7 and 14 ensure that $\varepsilon > 0$. □

Noting that $2\cos(\alpha)\cos(4\alpha) = \cos(5\alpha) - \cos(3\alpha)$ we see that choosing $\alpha \in (\alpha_*, 3\pi/10)$ such that $\cos(5\alpha)/\cos(3\alpha) \in \mathbb{R}\setminus\mathbb{Q}$ ensures that the map in Theorem 1 is an irrational rotation. In passing we note that α non-zero and algebraic forces $\cos(5\alpha)/\cos(3\alpha) \in \mathbb{R} \setminus \mathbb{Q}$, otherwise $\exp(\mathrm{i}\alpha)$ would be algebraic, which, since α was assumed to be non-zero and algebraic would contradict the Lindemann-Weierstrass Theorem (see, for example, [**2**]).

Thus, if $\alpha \in (\alpha_*, 3\pi/10)$ with $\cos(5\alpha)/\cos(3\alpha) \in \mathbb{R}\setminus\mathbb{Q}$, it follows that Lebesgue almost all initial values $x_0^{[1]}$ will give rise to a regular non-periodic orbit with initial condition $[1, \phi_*, x_0^{[1]}]$. By Lemma 16 the corresponding trajectory does not have bounces on the sides within a distance $\epsilon > 0$ of the tip of the triangle (when distance is measured along the bouncing side). Hence, the trajectory does not enter a small symmetric triangular region at the tip of the triangle and is thus not everywhere dense. A graphical illustration of this type of trajectory is shown in Figure 2.

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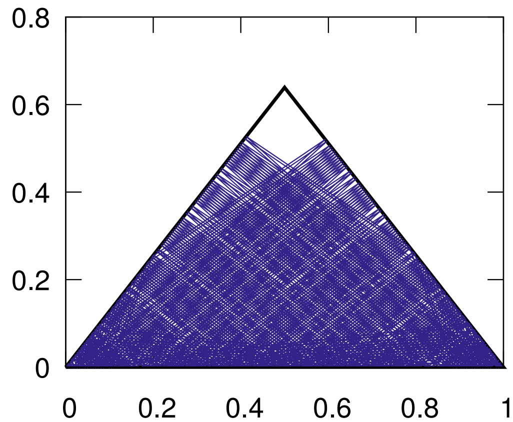

FIGURE 2. Finite trajectory of 500 bounces with initial condition on the base at $x_0^{[1]} = 1/\sqrt{2}$, $\phi_0 = 0.7329252\ldots$, see (4) in an isosceles triangle with inner angle $\alpha = \pi\sqrt{3}/6$ (see Lemma 3).

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