RIGGED HILBERT SPACES FOR CHAOTIC DYNAMICAL SYSTEMS

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Abstract. We consider the problem of rigging for the Koopman operators of the Renyi and the baker maps. We show that the rigged Hilbert space for the Renyi maps has some of the properties of a strict inductive limit and give a detailed description of the rigged Hilbert space for the baker maps.

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1. Introduction

The notion of a generalized spectral decomposition of selfadjoint operators on a Hilbert space goes back to Dirac [1], who assumed that a given selfadjoint operator A must be of the form

$$A = \int_{\sigma(A)} d\lambda \, \lambda \, |\lambda)(\lambda| \;\; , \tag{1}$$

where $\sigma(A)$ is the spectrum of the operator A. This formula is a straightforward generalization of the familiar decomposition of a selfadjoint operator on a finite-dimensional Hilbert space

$$A = \sum_i \lambda_i \; |e_i)(e_i| \; , \tag{2}$$

where λ_i and e_i are the eigenvalues and eigenvectors of A, respectively. In infinite dimensional Hilbert spaces, however, the situation is not so simple. The notion of an eigenvalue is replaced by the spectrum, but eigenvectors can be associated only with the discrete part of the spectrum. Nevertheless, a precise meaning can be given to the decomposition (2), if we replace eigenvectors by "generalized eigenvectors", which will in general lie outside the given Hilbert space. This is achieved by replacing the initial Hilbert space $\mathcal{H}$ by a dual pair $(\Phi, \Phi^\times)$, where Φ is a locally convex space, which is a dense subspace of $\mathcal{H}$ endowed with a topology, stronger than the Hilbert space topology. This procedure is referred to as rigging and the triple

$$\Phi \subset \mathcal{H} \subset \Phi^\times \tag{3}$$

is called a *rigged Hilbert space* (see [2–5] for details). Gelfand [3,4] was the first to give a precise meaning to the generalized eigenvectors, which was later elaborated by Maurin [5]. Although generalized eigenvectors have a very natural physical interpretation, generalized spectral decompositions have not been used in physics for a long time. Only a few papers had appeared by the end of the 60's (see, for example [6–8]), followed by a series of papers by Bohm an Gadella (see [2] and references therein). The latter publications are particularly significant, because they provide the basis for a rigorous and systematic approach to the problems of irreversibility and resonances in unstable quantum systems like the Friedrichs model [9]. The same ideas can be extended to chaotic dynamical systems, like Kolmogorov systems or exact systems [14,15]. The obsevable phase functions of dynamical systems evolve according to the Koopman operator [14]

$$Vf(x) = f(Sx),$$

where S is an endomorphism or an automorphism of a measure space, and f is a square-integrable phase function.

The spectrum of the Koopman operator determines the time scales of the approach to equilibrium very much in analogy with quantum unstable systems, where the spectra of the Hamiltonians determine the decay rates. More precisely, the eigenvalues of the Koopman operator or that of its adjoint, known as Frobenius-Perron operator, are the resonances of the power spectrum [24–27]. Eigenvalues and eigenvectors of simple chaotic systems have recently been constructed by several authors [10–13,26–33].

The question of the existence of a generalized spectral decomposition of extensions of the Koopman operator was raised and resolved by Antoniou and Tasaki [11–13]. This issue is delicate, because the original Gelfand-Maurin theory was constructed for operators which admit a spectral theorem [34], like normal operators, giving a generalized spectrum identical with the Hilbert space spectrum. The Koopman operator of unstable systems, however, either does not admit a spectral theorem, as in the case of exact systems [12], or the generalized spectrum is very different from the Hilbert space spectrum, as in the case of Kolomogorov systems [13].

The original Gelfand-Maurin theory had to be extended [11–13] to arbitrary Mackey topologies [35] associated with dual pairs $(\Phi, \Phi^{\times})$ of linear topological spaces.

Summarising for the reader's convenience, a dual pair $(\Phi, \Phi^{\times})$ of linear topological spaces constitutes a rigged Hilbert space for the linear endomorphism V of the Hilbert space $\mathcal{H}$ if the following conditions are satisfied:

1) Φ is a dense subspace of $\mathcal{H}$
2) Φ is complete and its topology is stronger than the one induced by $\mathcal{H}$
3) Φ is stable with respect to the adjoint $V^{\dagger}$ of V, i.e. $V^{\dagger}\Phi \subset \Phi$.
4) The adjoint $V^{\dagger}$ is continuous on Φ

The extension V_{ext} of V to the dual $\Phi^{\times}$ of Φ is then defined in the standard way as follows:

$$(\phi | V_{\text{ext}} f) = (V^{\dagger}\phi | f)$$

for every $\phi \in \Phi$.

In the sequel we shall not distinguish between V and V_{ext} if confusion is unlikely to arise.

The choice of the test function space Φ depends on the specific operator V and on the physically relevant questions to be asked about the system. For selfadjoint operators V, for example, the generalized spectral theorem can be justified for nuclear test function spaces; for normal operators this condition may be relaxed [36,37].

Here, we shall discuss the problem of rigging for the generalized spectral decompositions of the Koopman operators for two specific but typical models of chaotic systems, namely the Renyi maps and the baker maps.

In the case of the Renyi map various riggings exist [22] and our task will be to choose a tight rigging within spaces of analytic test functions. We call a rigging 'tight' if the test function space is the (set-theoretically) largest possible within a chosen family of test function spaces, such that the physically relevant spectral decomposition is meaningful. This notion of tightness is more general than that of Fredricks [7]. It turns out that the topology of this rigged Hilbert space enjoys some of the properties of a strict inductive limit of Banach spaces, which greatly simplifies convergence arguments.

The construction of the rigged Hilbert space for the baker map, on the other hand, reveals a different aspect of the problem of rigging. In fact, here the problem is to understand the very nature of the rigging, since the test function space is the tensor product of the space of polynomials with the space of square-integrable functions corresponding to the expanding and contracting fibres. Our task will be to investigate the properties of this rigged Hilbert space.

2. Rigged Hilbert spaces for the Renyi maps

In this section we discuss the rigged Hilbert spaces for the Koopman operator of the general β-adic Renyi map.

The β-adic Renyi map S on the interval [0,1) is the multiplication, modulo 1, by the integer $\beta \geq 2$:

$$S : [0,1) \to [0,1) \;\; : \qquad x \mapsto Sx = \beta x \pmod{1}. \, , \tag{1}$$

The probability densities $\rho(x)$ evolve according to the Frobenius–Perron operator U [15]:

$$U\rho(x) \equiv \sum_{y, S(y)=x} \frac{1}{|S'(y)|}\rho(y) = \frac{1}{\beta}\sum_{r=0}^{\beta-1} \rho\Big(\frac{x+r}{\beta}\Big) \; . \tag{2}$$

The Frobenius-Perron operator is a partial isometry on the Hilbert space L^2 of all square integrable functions over the unit interval; it is, moreover, the dual of the isometric Koopman operator V:

$$V\rho(x) = U^{\dagger}\rho(x) = \rho(Sx) \; . \tag{3}$$

In [12] two of us (I.A. and S.T.) constructed a spectral decomposition of the Koopman operator using a general algorithm based on the subdynamics decompositions.

The Koopman operator can be expressed as follows

$$V = \sum_{n=0}^{\infty} \frac{1}{\beta^n} \; |\tilde{B}_n)(B_n| \quad , \tag{4}$$

where $B_n(x)$ is the n-degree Bernoulli polynomial defined by the generating function [16, §9]:

$$\frac{ze^{zx}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n \tag{5}$$

and

$$|\widetilde{B}_n) \;\; = \begin{cases} |1) \; , & n = 0 \\ |\dfrac{(-1)^{(n-1)}}{n!}\{\delta^{(n-1)}(x-1) - \delta^{(n-1)}(x)\}) & n = 1, 2, \ldots \end{cases} \tag{6}$$

The bras $(.|$ and kets $|.)$ denote linear and antilinear functionals, respectively. Formula (4) defines a spectral decomposition for the Koopman and Frobenius-Perron operators in the following sense

$$(\rho|Vf) = (U\rho|f) = \sum_{n=0}^{\infty} \frac{1}{\beta^n} (\rho|\tilde{B}_n) \; (B_n|f),$$

for any density function ρ and obsevable f in the appropriate pair $(\Phi, \Phi^{\times})$. Consequently, the Frobenius-Perron operator acts on density functions as

$$U\rho(x) = \int_0^1 dx' \rho(x') + \sum_{n=1}^{\infty} \frac{\rho^{(n-1)}(1) - \rho^{(n-1)}(0)}{n!\beta^n} B_n(x) \; . \tag{7}$$

The orthonormality of the system $|\tilde{B}_n)$ and $(B_n|$ follows immediately, while the completeness relation is just the Euler–MacLaurin summation formula for the Bernoulli polynomials [16, §9]

$$\rho(x) = \int_0^1 dx' \rho(x') + \sum_{n=1}^{\infty} \frac{\rho^{(n-1)}(1) - \rho^{(n-1)}(0)}{n!} B_n(x) \; . \tag{8}$$

The Bernoulli polynomials are the only polynomial eigenfunctions as any polynomial can be uniquely expressed as a linear combination of the Bernoulli polynomials.

The spectral decomposition (4) has no meaning in the Hilbert space L^2, as the derivatives $\delta^{(n)}(x)$ of Dirac's delta function appear as right eigenvectors of V. A natural way to give meaning to formal eigenvectors of operators which do not admit eigenvectors in Hilbert space is to extend the operator to a suitable rigged Hilbert space. A suitable test function space is the space $\mathcal{P}$ of polynomials. The space $\mathcal{P}$ fulfills conditions 1–4:

1) $\mathcal{P}$ is dense in L^2 (see [17, ch.15]),
2) $\mathcal{P}$ is a nuclear LF-space [17, ch.51] and thus, complete and barreled,
3) $\mathcal{P}$ is stable with respect to the Frobenius-Perron operator U, and
4) U is continuous with respect to the topology of $\mathcal{P}$, because U preserves the degree of polynomials.

It is, therefore, an appropriate rigged Hilbert space, which gives meaning to the spectral decomposition of V.

We shall, however, look for a tight rigging. The test functions should at least provide a domain for the Euler-MacLaurin summation formula (8). The requirement of absolute convergence of the series (8) means that

$$\sum_{n=1}^{\infty} \Big| \frac{\phi^{(n-1)}(y)}{n!} B_n(x) \Big| < \infty \ . \qquad (y = 0, 1) \tag{9}$$

This implies [12] that the appropriate test functions are restrictions on [0,1) of entire functions of exponential type c with $0 < c < 2\pi$. For simplicity we identify the test functions space with the space $\mathcal{E}_c$ of entire functions $\phi(z)$ of exponential type $c > 0$ such that

$$|\phi(z)| \le K e^{c|z|} \ , \quad \forall z \in \mathbf{C}, \text{ for some } K > 0 \ . \tag{10}$$

Each member of the whole family $\mathcal{E}_c$, $0 < c < 2\pi$ is a suitable test function space, since properties 1–4 are fulfilled. Indeed, each space $\mathcal{E}_c$ is a Banach space with norm [17, ch.22]:

$$\|\phi\|_c \equiv \sup_{z\in\mathbf{C}} \left|\phi(z)\right| e^{-c|z|} \ , \tag{11}$$

which is dense in the Hilbert space L^2, as $\mathcal{E}_c$ includes the polynomial space $\mathcal{P}$. Each $\mathcal{E}_c$ is stable under the Frobenius-Perron operator U, and it is easily verified that U is continuous on $\mathcal{E}_c$. Now, observe that the spaces are ordered:

$$\mathcal{E}_c \subset \mathcal{E}_{c'} \ , \qquad c < c',$$

and consider the space

$$\tilde{\mathcal{E}}_{2\pi} \equiv \bigcup_{c<2\pi} \mathcal{E}_c \ . \tag{12}$$

The space $\tilde{\mathcal{E}}_{2\pi}$, also preserved by U, is the (set-theoretically) largest test function space in our case. Since $\tilde{\mathcal{E}}_{2\pi}$ is a natural generalization of the space $\mathcal{P}$ of polynomials, we want to equip it with a topology which is a generalization of the topology of $\mathcal{P}$.

Recall that $\mathcal{P}$ was given the strict inductive limit topology of the spaces $\mathcal{P}^n$ of all polynomials of degree $\leq n$. A very important property of this topology is that the strict inductive limit of complete spaces is complete. Moreover, it is exceptionally simple to describe convergence in this topology. For example, a sequence $\{w_n\}$ of polynomials converges in $\mathcal{P}$ if and only if the degrees of all w_n are uniformly bounded by some n_0 and $\{w_n\}$ converges in $\mathcal{P}^{n_0}$.

We cannot, however, define the strict inductive topology on $\tilde{\mathcal{E}}_{2\pi}$, because for $c < c'$ the topology on $\mathcal{E}_c$ induced by $\mathcal{E}_{c'}$ is essentially stronger than the initial one. Nevertheless, as we shall see in the theorem below, it is possible to define a topology on $\tilde{\mathcal{E}}_{2\pi}$, which is a natural extension of the topology on $\mathcal{P}$ in the following sense:

Theorem. *There is a locally convex topology $\mathcal{T}$ on $\tilde{\mathcal{E}}_{2\pi}$ for which it is a nuclear, complete Montel space. Moreover, a sequence $\{f_n\} \subset \tilde{\mathcal{E}}_{2\pi}$ is convergent in the $\mathcal{T}$ topology if and only if there is $c_0 \in (0, 2\pi)$ such that*

1° $f_n,\ n = 1, 2, \ldots$, *are of exponential type* c_0

2° $\{f_n\}$ *converges in* $\|\cdot\|_{c_0}$ *- norm.*

Proof. Denote by $\hat{f}$ the Fourier transform of a function f and by $\check{f}$ its converse. By Schwartz's extension of the Paley-Wiener Theorem [19, vol.II, p.106] a function f belongs to $\mathcal{E}_c$ if and only if $\hat{f}$ is a distribution with compact support contained in the interval $[-c, c]$.

Note that, if the function $f \in \tilde{\mathcal{E}}_{2\pi}$ is integrable or square integrable then $\hat{f}$ is a function. However, for an arbitrary function its Fourier transform is correctly defined only as a distribution with compact support, i.e. as a continuous linear functional on the space $C^\infty(\Omega)$ of all infinitely differentiable functions on the interval $\Omega = (-2\pi, 2\pi)$, endowed with the topology of uniform convergence on compact subsets of Ω, of functions together with all their derivatives.

The Fourier transform, therefore, establishes an isomorphism between $\tilde{\mathcal{E}}_{2\pi}$ and the topological dual $C^\infty(\Omega)^\times$ of the space $C^\infty(\Omega)$.

Consequently, the strong dual topology of $C^\infty(\Omega)^\times$ can be transported through the inverse Fourier transform to the space $\tilde{\mathcal{E}}_{2\pi}$. The strong dual topology is the topology of uniform convergence on bounded subsets of $C^\infty(\Omega)$. Then $C^\infty(\Omega)^\times$ is nuclear [17, p.530], complete [19, vol.I, p.89] and a Montel space [17, prop. 34.4 and 36.10]. In this way we obtain on $\tilde{\mathcal{E}}_{2\pi}$ a topology with the same properties.

We shall now prove the second part of the theorem. Let $\{f_n\}$ be convergent to zero in $\tilde{\mathcal{E}}_{2\pi}$. This means that $\{\check{f}_n\}$ converges in $C^\infty(\Omega)^\times$. Therefore $\{\check{f}_n\}$ is a bounded subset of $C^\infty(\Omega)^\times$, which implies [17, th. 34.4, p.359] that the supports of all $\check{f}_n$ are contained in a compact set $K \subset \Omega$.

Take c with $c < 2\pi$ and $K \subset (-c, c)$. Therefore (see [19, vol.I, th.XXVI] and the remark afterwards which remain true if we replace $\mathbf{R^1}$ by the open set $\Omega = (-2\pi, 2\pi)$), there is a number $p \geq 0$ and a family of continuous functions $g_{j,n}$ such that the supports of $g_{j,n}$ are contained in the interval $(-c, c)$,

$$\check{f}_n = \sum_{j \leq p} D^j g_{j,n} \tag{13}$$

(D^j denotes the j-th derivative, classical or in the sense of distributions) and $g_{j,n}(x)$ converges uniformly to zero as $n \to \infty$.

Using the above representation of $\check{f}_n$ we obtain that $\check{f}_n$ converges to zero uniformly on each set U_A

$$U_A \equiv \{f \in C^\infty(\Omega) : \sup_{x \in [-c,c]} |\frac{d^j}{dx_j} f(x)| \leq A, \quad j = 0, 1, \ldots, p\} \ , \tag{14}$$

where $A > 0$. Indeed, for each j

$$\begin{aligned} |\langle D^j g_{j,n}, f\rangle| &= |(-1)^j \int_{-c}^{c} g_{j,n}(x) \frac{d^j}{dx_j} f(x) \ dx| \\ &\leq A \int_{-c}^{c} |g_{j,n}(x)| \ dx \to 0 \ , \end{aligned}$$

as $n \to \infty$.

Let us take any $c_0 \in (c, 2\pi)$. Then for each $z \in \mathbf{C}$ the function

$$x \longmapsto e^{izx} e^{-c_0|z|}, \quad |x| \leq c \ , \tag{15}$$

belongs to U_A. Indeed

$$\begin{aligned} |\frac{d^j}{dx_j}(e^{izx}e^{-c_0|z|})| &\leq |z|^j e^{|z|(|x|-c_0)} = |z|^j e^{|z|(|x|-c)} e^{-(c_o-c)|z|} \\ &\leq |z|^j e^{-(c_o-c)|z|} \ . \end{aligned}$$

The right hand side is bounded, for each $j = 0, 1, ..., p$, by some constant A_j. Thus taking

$$A = \max_{0 \leq j \leq p} A_j \tag{16}$$

we see that the functions (15) belong to U_A, for each $z \in \mathbf{C}$.

From

$$f_n(z) = (\check{f}_n)\hat{}\,(z) \tag{17}$$

and uniform convergence of $\check{f}_n$ on U_A we have

$$\sup_{z \in \mathbf{C}} |f_n(z)| e^{-c_0|z|} = \sup_{z \in \mathbf{C}} |\langle \check{f}_n, e^{iz\cdot} e^{-c_0|z|} \rangle| \longrightarrow 0 \ ,$$

as $n \to \infty$, whaich means that

$$||f_n||_{c_0} \longrightarrow 0 \ .$$

This proves 2^{o}. Condition 1^{o} is also satisfied because we have chosen $c_0 > c$. Thus, the supports of the $\check{f}_n$s are also contained in $(-c_0, c_0)$ and by the Paley-Wiener-Schwartz theorem the f_ns are of exponential type c_0.

The converse of the second part of the theorem is now trivial. If $\{f_n\}$ satisfies 1^{o} and 2^{o} then by applying the Paley-Wiener-Schwartz theorem again we obtain convergence of $\{\hat{f}_n\}$ in $C^\infty(\Omega)^\times$.

Remarks

1. Using the above method one can show an analogous criterion of convergence for bounded nets in $\tilde{\mathcal{E}}_{2\pi}$ but not for an arbitrary net.

2. Note that it is not always possible to obtain convergence of the type given in the above theorem. Actually, to prove the second part of the theorem we needed the following property:

Let F be a Frechet space and let $\{x'_n\}$ be a sequence in its dual F' which converges to zero in the strong dual topology. Then there exists an open subset U of F such that

$$|\langle x'_n, x\rangle| \longrightarrow 0 \ , \quad \text{uniformly for } x \in U \ . \tag{18}$$

As mentioned in [20], some concrete F-spaces have this property although it is not true in general. It was stated there as an open problem to describe those F-spaces for which (18) is true. This situation motivated us to include the full proof.

3. An alternative, but less constructive proof of the theorem can be found in [22]. It is based on a theorem by Raikov [23] and the nuclearity of the imbedding

$$\mathcal{E}_c \hookrightarrow \mathcal{E}_{c'} \ , \qquad c < c'.$$

3. The rigged Hilbert space for the baker transformations

The β-adic, $\beta = 2, 3, \cdots$, baker's transformation B on the unit square $Y = [0,1)\times[0,1)$ is a two-step operation: 1) squeeze the 1×1 square to a $\beta \times 1/\beta$ rectangle and 2) cut the rectangle into β $(1 \times 1/\beta)$-rectangles and pile them up to form another 1×1 square:

$$(x,y) \longmapsto B(x,y) = (\beta x - r, \frac{y+r}{\beta}) \quad (\text{for } \frac{r}{\beta} \le x < \frac{r+1}{\beta},\ r = 0, \cdots \beta - 1) \ . \quad (19)$$

The invariant measure of the β-adic baker transformation is the Lebesgue measure on the unit square. The probability densities $\rho(x,y)$ evolve according to the Frobenius-Perron operator U [15]:

$$\begin{aligned} U\rho(x,y) &\equiv \rho(B^{-1}(x,y)) \\ &= \rho\Big(\frac{x+r}{\beta}, \beta y - r\Big) \ , \quad (\text{for } \frac{r}{\beta} \le y < \frac{r+1}{\beta},\ r = 0, \cdots \beta - 1) \ , \quad (20) \end{aligned}$$

The Frobenius-Perron and Koopman operators are unitary on the Hilbert space $L^2 = L^2_x \otimes L^2_y$ of square integrable densities over the unit square and has countably degenerate Lebesgue spectrum on the unit circle plus the simple eigenvalue 1 associated with the equilibrium (as is the case for all Kolmogorov automorphisms).

The β-adic baker automorphism B is the natural extension [18] of the β-adic Renyi map on the unit interval [0,1), described in the previous section.

The Koopman operator V has a spectral decomposition involving Jordan blocks, which was obtained [13] using a generalized iterative operator method based on subdynamics:

$$V = \ |\tilde{f}_{00})(f_{00}| \ + \sum_{\nu=1}^{\infty} \Big\{ \sum_{r=0}^{\nu} \frac{1}{\beta^\nu} \ |\tilde{f}_{\nu,r})(f_{\nu,r}| \ + \sum_{r=0}^{\nu-1} \ |\tilde{f}_{\nu,r+1})(f_{\nu,r}| \ \Big\} \ , \qquad (21)$$

The vectors $|\tilde{f}_{\nu,r})$ and $(f_{\nu,r}|$ form a Jordan basis

$$(f_{\nu,r}|\ V = \begin{cases} \dfrac{1}{\beta^\nu}\ (f_{\nu,r}|\ + (f_{\nu,r+1}| & (r=0,\ldots,\nu-1) \\ \dfrac{1}{\beta^\nu}\ (f_{\nu,r}| & (r=\nu) \end{cases} \tag{22}$$

$$V\ |\tilde{f}_{\nu,r}) = \begin{cases} \dfrac{1}{\beta^\nu}\ |\tilde{f}_{\nu,r}) +\ |\tilde{f}_{\nu,r-1}) & (r=1,\ldots,\nu) \\ \dfrac{1}{\beta^\nu}\ \ |\tilde{f}_{\nu,r}) & (r=0) \end{cases} \tag{23}$$

$$(f_{\nu,r}|\ \tilde{f}_{\nu',r'}\rangle = \delta_{\nu\nu'}\delta_{rr'}\ , \tag{24}$$

$$\sum_{\nu=0}^{\infty}\sum_{r=0}^{\nu}\ |\tilde{f}_{\nu,r})(f_{\nu,r}|\ = I\ . \tag{25}$$

While the Koopman operator V is unitary in the Hilbert space L^2 and thus has spectrum on the unit circle $|z| = 1$ in the complex plane, the spectral decomposition (21) includes the numbers $1/\beta^\nu < 1$ which are not in the Hilbert space spectrum. The spectral decomposition (21) also shows that the Frobenius–Perron operator has Jordan-block parts despite the fact that it is diagonalizable in the Hilbert space. As the left and right principal vectors contain generalized functions, the spectral decomposition (21) has no meaning in the Hilbert space L^2.

It was shown in [13] that the principal vectors $f_{\nu,j}$ and $\tilde{f}_{\nu,j}$ are linear functionals over the spaces $L^2_x \otimes \mathcal{P}_y$ and $\mathcal{P}_x \otimes L^2_y$, respectively. Therefore, our purpose is to define an appropriate topology on these spaces. We shall give the construction for $\mathcal{P}_x \otimes L^2_y$ only; a similar argument applies to $L^2_x \otimes \mathcal{P}_y$

We will start from the most natural, i.e. the strict inductive limit topology, which coincides with other, apparently stronger, tensor product topologies.

Let us consider the space $\mathcal{P} \otimes L^2$ (for simplicity we omit the subscripts x and y), where $\mathcal{P}$ is the space of all polynomials w of finite degree:

$$w = \sum_{k=0}^{n} a_k x^k\ , \tag{26}$$

L_y^2 is the space of all square integrable functions on the space $Y = [0, 1]$.

Let $\mathcal{P}^n$ be the space of all polynomials of degree $\leq n$. For a w of the form (26)

$$\|w\|_n = \max_{0 \leq k \leq n} |a_k| \tag{27}$$

defines a norm on $\mathcal{P}^n$. Then $\mathcal{P}$ is defined as the union $\bigcup_n \mathcal{P}^n$ with the strict inductive limit topology [17, sec.13].

Similarly $\mathcal{P} \otimes L^2$ was defined in [13] as the strict inductive limit of the spaces $\mathcal{P}^n \otimes L^2$ endowed with the topology τ, generated by the norms

$$\|\sum_{k=1}^{n} x^k \otimes \phi_k\|_n = \max_{0 \leq k \leq n} \|\phi\|_{L^2} \ . \tag{28}$$

We may, therefore, write symbolically

$$\mathcal{P} \otimes L^2 = \varinjlim_{n} (\mathcal{P}^n \otimes L^2) \ . \tag{29}$$

It is very easy to see that

$$(\bigcup_n \mathcal{P}^n) \otimes L^2 = \bigcup_n (\mathcal{P}^n \otimes L^2) \ . \tag{30}$$

Thus, we have *algebraically*:

$$(\varinjlim_{n} \mathcal{P}^n) \otimes L^2 = \varinjlim_{n} (\mathcal{P}^n \otimes L^2) \ , \tag{31}$$

and, as we will see below, also topologically.

If fact, it will be shown that the τ-topology defined by the seminorms (28) is also, roughly speaking, the only natural locally convex tensor product topology on the space $\mathcal{P}^n \otimes L^2$).

This will be proved in three steps:

1) τ is a cross-seminorm topology, i.e. the tensor product seminorm (28) is a cross-norm [21] when restricted to the Banach spaces $\mathcal{P}^n$ and L^2. Indeed, for $w \in \mathcal{P}^n \ , \ \ f \in L^2$

$$
\begin{aligned}
\|w \otimes f\|_n &= \|\sum_{k=1}^{n} a_k x^k \otimes f\|_n = \|\sum_{k=1}^{n} x^k \otimes (a_k f)\|_n \\
&= \max_{0\le k\le n} \|a_k f\|_{L^2} = \max_{0\le k\le n} |a_k| \ \|f\|_{L^2} \\
&= \|w\|_n \|f\|_{L^2}
\end{aligned}
$$

2) The τ-topology is weaker than the projective topology (shortly π-topology) on the tensor product $\mathcal{P}\otimes L^2$) (see [17, sec. 43] for the definition). Indeed, consider an element

$$
\tilde{f} = \sum_{i=1}^{N} w_i \otimes f_i \tag{29}
$$

of the space $\mathcal{P} \otimes L^2$) and let n be the maximal degree of $w_i,\ i = 1, ..., N$. Then we can write

$$
w_i = \sum_{k=0}^{N} a_k^{(i)} x^k \ , \quad i = 1, ..., N \tag{30}
$$

(some $a_k^{(i)}$ can be zero).

Therefore

$$
\begin{aligned}
\tilde{f} &= \|\sum_{i=1}^{N} w_i \otimes f_i\|_n = \|\sum_{i=1}^{N} (\sum_{k=0}^{n} a_k^{(i)} x^k) \otimes f_i\|_n \\
&= \|\sum_{k=o}^{n} (\sum_{i=1}^{N} a_k^{(i)} f_i) \otimes x^k\|_n = \max_{0\le k\le n} \|\sum_{i=1}^{N} a_k^{(i)} f_i\|_{L^2} \\
&\le \max_{0\le k\le n} \sum_{i=1}^{N} |a_k^{(i)}| \ \|f_i\|_{L^2} \ .
\end{aligned}
$$

Denote by R the right hand side in the above inequalities and let k_0 be the index which realizes the maximum of R. Then

$$
\begin{aligned}
R = \sum_{i=1}^{N} |a_{k_0}^{(i)}| \; \|f_i\|_{L^2} &\leq \sum_{i=1}^{N} \max_{0\leq k\leq n} |a_k^{(i)}| \; \|f_i\|_{L^2} \\
&= \sum_{i=1}^{N} \|w_i\|_n \|f_i\|_{L^2} \ .
\end{aligned}
$$

Therefore, we obtain the inequality

$$
\|\tilde{f}\|_n \leq \sum_{i=1}^{N} \|w_i\|_n \|f_i\|_{L^2} \ , \tag{31}
$$

which does not depend on the particular representation of $\tilde{f}$. Thus

$$
\|\tilde{f}\|_n \leq \|\tilde{f}\|_{\pi,n} \ , \tag{32}
$$

where $\|\cdot\|_{\pi,n}$ denotes the π-seminorm corresponding to the seminorm $\|\cdot\|_n$ on $\mathcal{P}$ and $\|\cdot\|_{L^2}$.

3) the τ-topology is stronger than the ε-topology (see [17 ,sec.43] for the definition) on $\mathcal{P}\otimes L^2$. To see this let us first note that the space $\mathcal{P}$ with its topology can be identified with the space $C_c^0(X)$ of continuous functions on X with compact support [17, p.132], provided we take as the locally compact space the set $\mathbf{N}\cup\{0\}$ with the discrete topology. In such a case, functions with compact support are just sequences with at most finitely many non-zero elements and the family of seminorms is here precisely the same as that for $\mathcal{P}$ described above. Similarly, $\mathcal{P}\otimes L^2$ with the τ-topology can be identified with $C_c^0(X;L^2)$ which is a subspace of the space $C^0(X;L^2)$ of all continuous functions on X with values in L^2 (see [17, p.412] for the definition of the topology).

On the other hand, $C_c^0(X;L^2)$ is topologically isomorphic with $C_c^0(X)\hat{\otimes}_\varepsilon L^2$ ($\hat{}$ - denotes completion). Since the topology induced by $C^0(X;L^2)$ on $C_c^0(X;L^2)$ is weaker than the τ-topology we obtain that:

the ε-topology on $\mathcal{P} \otimes L^2$ is weaker than τ-topology.

Therefore after completion we obtain

$$\mathcal{P}\hat{\otimes}_\pi L^2 \subset \mathcal{P}\hat{\otimes}_\tau L^2 \subset \mathcal{P}\hat{\otimes}_\varepsilon L^2 \ .$$

However, since $\mathcal{P}$ is a nuclear space we have [17, th. 50.1]

$$\mathcal{P}\hat{\otimes}_\varepsilon L^2 \doteq \mathcal{P}\hat{\otimes}_\pi L^2 \tag{33}$$

($\doteq$ denotes topological isomorphism). Because $\mathcal{P} \otimes L^2$ is already complete in the τ-topology [13] we obtain that:

the τ-topology on $\mathcal{P} \otimes L^2$ coincides with the ε and the π topology.

4. Concluding remarks

1. We have characterized the natural rigged Hilbert spaces of analytic functions associated with the prototype of dynamical systems, namely the Renyi and the baker transformations. In the case of the Renyi, map we constructed a tight rigged Hilbert space $\tilde{\mathcal{E}}_{2\pi}$ within the spaces of analytic functions, which gives meaning to the simple resonance spectrum. We have shown that $\tilde{\mathcal{E}}_{2\pi}$ inherits the crucial properties of strict inductive limits of Banach spaces without being a strict inductive limit itself. For the baker maps we characterized the topology of the tensor product which gives meaning to the multiple resonance spectrum.
2. We expect that these rigged Hilbert space topologies are typical for chaotic maps, if the evolution of analytic densities is considered. In the case of non-analytic densities we may have test function spaces satisfying properties 1–4 with different extension properties.

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