RESOLVENT ESTIMATES FOR OPERATORS BELONGING TO EXPONENTIAL CLASSES

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Abstract. For $a, \alpha > 0$ let $E(a, \alpha)$ be the set of all compact operators A on a separable Hilbert space such that $s_n(A) = O(\exp(-an^\alpha))$, where $s_n(A)$ denotes the n-th singular number of A. We provide upper bounds for the norm of the resolvent $(zI - A)^{-1}$ of A in terms of a quantity describing the departure from normality of A and the distance of z to the spectrum of A. As a consequence we obtain upper bounds for the Hausdorff distance of the spectra of two operators in $E(a, \alpha)$.

1. Introduction

Let A and B be compact operators on a Hilbert space. It is known that if $\|A - B\|$ is small then the spectra of A and B are close in a suitable sense (for example, with respect to the Hausdorff metric on the space of compact subsets of $\mathbb{C}$). Just how close are they? Standard perturbation theory gives bounds in terms of quantities that require a rather detailed knowledge of the spectral properties of both operators, for example the norms of the resolvents of A and B along contours in the complex plane, which are difficult to obtain in practice.

The main concern of this article is to derive an upper bound for the norm of the resolvent $(zI - A)^{-1}$ of an operator A belonging to certain subclasses of compact operators in terms of simple, readily computable quantities, typically involving the distance of z to the spectrum of A and a number measuring the departure from normality of A. As a result, we obtain simple upper bounds for the Hausdorff distance of the spectra of two operators in these subclasses. Estimates of this type have previously been obtained for operators in the Schatten classes (see [Gil, Ban]) and more generally (but less sharp), for operators belonging to Φ-ideals (see [Pok]).

These subclasses, termed *exponential classes*, are constructed as follows. For $a, \alpha > 0$ let $E(a, \alpha)$ denote the collection of all compact operators on a separable Hilbert space for which $s_n(A) = O(\exp(-an^\alpha))$, where $s_n(A)$ denotes the n-th singular number of A. As we shall see, $E(a, \alpha)$ is not a linear space (see Remark 2.9), hence *a fortiori* not an operator ideal, and may thus be viewed as a slightly pathological object in this context. There is nevertheless compelling reason to consider these classes: on the one hand, the resolvent bounds given in [Ban, Gil, Pok], while applicable to operators in $E(a, \alpha)$, can be improved significantly (see Remark 3.2), the lack of linear structure posing almost no problem for the derivation of these improvements. On the other hand, operators belonging to exponential classes arise

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Date: December 20, 2006.

2000 *Mathematics Subject Classification.* Primary 47A10; Secondary 47B06, 47B07.

Key words and phrases. Resolvent growth, exponential class, departure from normality, bounds for spectral distance.

naturally in a number of different ways. For example, if A is an integral operator with real analytic kernel given as a function on $[0,1]^d \times [0,1]^d$, then $A \in E(a, 1/d)$ for some $a > 0$ (see [KR]). Other examples of operators in the exponential class $E(a, 1/d)$ for some $a > 0$ include composition operators on Bergman spaces over domains in $\mathbb{C}^d$ whose symbols are strict contractions, or more generally transfer operators corresponding to holomorphic map-weight systems on $\mathbb{C}^d$, the latter providing one of the motivations to look more closely into the properties of operators belonging to exponential classes (see Example 2.3 (iv) and [BanJ1, BanJ2]).

This article is organised as follows. In Section 2 we define the exponential classes and study some of their properties. In particular, we shall give a sharp description of the behaviour of exponential classes under addition (see Proposition 2.8), and a sharp characterisation of the eigenvalue asymptotics of an operator in a given exponential class (see Proposition 2.10). Some of the arguments in this section rely on results concerning monotonic arrangements of sequences, which are presented in the Appendix. In Section 3 we will use techniques similar to those already employed in [Ban] to obtain resolvent estimates for operators in $E(a, \alpha)$ (see Theorem 3.13). In particular we shall give a sharp estimate for the growth of the resolvent of a quasi-nilpotent operator in $E(a, \alpha)$ (see Proposition 3.1). These estimates will then be used in the final section to deduce Theorem 4.2, which provides spectral variation and spectral distance formulae for operators in $E(a, \alpha)$.

Notation 1.1. Throughout this article H and H_i will be assumed to be separable Hilbert spaces. We use $L(H_1, H_2)$ to denote the Banach space of bounded linear operators from H_1 to H_2 equipped with the usual norm and $S_\infty(H_1, H_2) \subset L(H_1, H_2)$ to denote the closed subspace of compact operators from H_1 to H_2. We shall often write L or S_∞ if the Hilbert spaces H_1 and H_2 are understood.

For $A \in S_\infty(H_1, H_2)$ we use

$$s_k(A) := \inf \{ \|A - F\| \,:\, F \in L(H_1, H_2),\, \operatorname{rank}(F) < k \} \quad (k \in \mathbb{N})$$

to denote the *k-th approximation number* of A, and $s(A)$ to denote the sequence $\{s_k(A)\}_{k=1}^\infty$.

The spectrum and the resolvent set of $A \in L(H, H)$ will be denoted by $\sigma(A)$ and $\varrho(A)$, respectively. For $A \in S_\infty(H, H)$ we let $\lambda(A) = \{\lambda_k(A)\}_{k=1}^\infty$ denote the sequence of eigenvalues of A, each eigenvalue repeated according to its algebraic multiplicity, and ordered by magnitude, so that $|\lambda_1(A)| \geq |\lambda_2(A)| \geq \ldots$. Similarly, we write $|\lambda(A)|$ for the sequence $\{|\lambda_k(A)|\}_{k=1}^\infty$.

We note that the approximation numbers coincide with the *singular numbers*, that is,

$$s_k(A) = \sqrt{\lambda_k(A^* A)} \quad (k \in \mathbb{N}),$$

where $A^* \in L(H_2, H_1)$ denotes the adjoint of $A \in L(H_1, H_2)$. For more information about these notions see, for example, [Pie, GK, DS, Rin].

2. Exponential classes

Exponential classes arise by grouping together all operators A whose singular numbers $s_n(A)$ decay at a given (stretched) exponential rate, that is, $s_n(A) = O(\exp(-a n^\alpha))$ for fixed $a > 0$ and $\alpha > 0$. Our main concern in this section will be to investigate how these classes behave under addition and multiplication, and to determine the rate of decay of the eigenvalue sequence of an operator in a given class. Some of the arguments in this section depend on results concerning monotonic arrangements of sequences, which are discussed in the Appendix.

Definition 2.1. Let $a > 0$ and $\alpha > 0$. Then

$$\mathcal{E}(a,\alpha) := \left\{ x \in \mathbb{C}^{\mathbb{N}} \; : \; |x|_{a,\alpha} := \sup_{n\in\mathbb{N}} |x_n| \exp(an^{\alpha}) < \infty \right\},$$

and

$$E(a,\alpha;H_1,H_2) := \left\{ A \in S_{\infty}(H_1,H_2) \; : \; |A|_{a,\alpha} := \sup_{n\in\mathbb{N}} s_n(A) \exp(an^{\alpha}) < \infty \right\}.$$

are called *exponential classes of type* (a,α) *of sequences* and *operators*, respectively. The numbers $|x|_{a,\alpha}$ and $|A|_{a,\alpha}$ are called (a,α)-*gauge* or simply *gauge* of x and A, respectively.

Whenever the Hilbert spaces are clear from the context, we suppress reference to them and simply write $E(a,\alpha)$ instead of $E(a,\alpha;H_1,H_2)$.

Remark 2.2. Note that $\mathcal{E}(a,\alpha)$ is a Banach space when equipped with the gauge $|\cdot|_{a,\alpha}$. On the other hand, the set $E(a,\alpha)$, the non-commutative analogue of $\mathcal{E}(a,\alpha)$, is not even a linear space in general (see Proposition 2.8 and Remark 2.9 below). The reason for this is that if a sequence lies in $\mathcal{E}(a,\alpha)$ then a rearrangement of this sequence need not; in particular $\mathcal{E}(a,\alpha)$ is not a Calkin space in the sense of [Sim2, p. 26] (cf. also [Cal]). However, $E(a,\alpha;H,H)$ turns out to be a *pre-ideal* (see Remark 2.6).

Operators belonging to exponential classes arise naturally in a number of different contexts.

Example 2.3.
(i) Let σ be a complex measure on the circle group $\mathcal{T}$ such that its Fourier transform satisfies

$$|\widehat{\sigma}(n)| \leq \exp(-a|n|) \qquad (n \in \mathbb{Z}).$$

It is not difficult to see that this is the case if and only if σ is absolutely continuous with respect to Haar measure on $\mathcal{T}$ and the corresponding Radon-Nikodým derivative is holomorphic on $\mathcal{T}$.

Let $L^2(\mathcal{T})$ be the complex Hilbert space of square-integrable functions on $\mathcal{T}$, with respect to Haar measure on $\mathcal{T}$. Let $A : L^2(\mathcal{T}) \longrightarrow L^2(\mathcal{T})$ be the convolution operator

$$Af = f * \sigma .$$

The spectrum of A is $\widehat{\sigma}(\mathbb{Z}) \cup \{0\}$ and the spectrum of A^*A equals $\widehat{\sigma * \widetilde{\sigma}}(\mathbb{Z}) \cup \{0\} = |\widehat{\sigma}(\mathbb{Z})| \cup \{0\}$, where $d\widetilde{\sigma}(t) = d\sigma(t^{-1})$ (cf. [BerF, p. 87]). Moreover, A is a compact operator and the non-zero eigenvalues of A are precisely the numbers $\widehat{\sigma}(n)$ for $n \in \mathbb{Z}$. In order to locate A in the scale of exponential classes, we enumerate the eigenvalues of A as follows

$$x_n = \widehat{\sigma}\left(\frac{(-1)^n}{4}(2n + (-1)^n - 1) \right) \qquad (n \in \mathbb{N}).$$

Then the sequence x belongs to the class $\mathcal{E}(a/2,1)$ with $|x|_{1/2,1} \leq \exp(a/2)$ since

$$\begin{aligned} & \left|\widehat{\sigma}\left(\frac{(-1)^n}{4}(2n+(-1)^n-1)\right)\right| \\ \leq\ & \exp\left(-a\left|\frac{(-1)^n}{4}(2n+(-1)^n-1)\right|\right) \\ \leq\ & \exp\left(-\frac{a}{2}(n-1)\right) = \exp\left(\frac{a}{2}\right)\exp\left(-\frac{a}{2}n\right). \end{aligned}$$

By Corollary 5.4 the decreasing arrangement $x^{(+)}$ of x also belongs to $\mathcal{E}(a/2,1)$ with $|x^{(+)}|_{1/2,1} \leq \exp(a/2)$. Thus $s(A) = |\lambda(A)| \in \mathcal{E}(a/2,1)$ and hence $A \in E(a/2,1)$ with $|A|_{a/2,1} \leq \exp(a/2)$.

(ii) A variant of the above example is discussed by König and Richter [KR], who showed that if A is an integral operator on the space of Lebesgue square-integrable functions on the d-dimensional unit-cube $[0,1]^d$ whose kernel is real analytic on $[0,1]^d \times [0,1]^d$, then $A \in E(1/d)$.

(iii) For a domain $\Omega \subset \mathbb{R}^d$ with $d > 1$ let $h^2(\Omega)$ be the Bergman space of Lebesgue square-integrable harmonic functions on Ω. If $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ are two domains such that Ω_2 is *compactly contained* in Ω_1, that is $\overline{\Omega_2}$ is a compact subset of Ω_1, then the canonical embedding $J : h^2(\Omega_1) \hookrightarrow h^2(\Omega_2)$ given by $Jf = f|\Omega_2$ belongs to the exponential class $E(1/(d-1))$. Moreover, for domains Ω_1 and Ω_2 with simple geometries it is possible to sharply locate J in an exponential class $E(a, 1/(d-1))$ and calculate the corresponding $(a, 1/(d-1))$-gauge of J exactly. See [BanC].

(iv) For $\Omega \subset \mathbb{C}^d$ a bounded domain, let $L^2_{Hol}(\Omega)$ denote the Bergman space of holomorphic functions which are square-integrable with respect to $2d$-dimensional Lebesgue measure on Ω. Given a collection $\{\phi_1, \ldots, \phi_K\}$ of holomorphic maps $\phi_k : \Omega \to \Omega$ and a collection $\{w_1, \ldots, w_K\}$ of bounded holomorphic functions $w_k : \Omega \to \mathbb{C}$ consider the corresponding linear operator A on $L^2_{Hol}(\Omega)$ given by

$$Af := \sum_{k=1}^{K} w_k \cdot f \circ \phi_k\,.$$

If $\cup_k \phi_k(\Omega)$ is compactly contained in Ω (see the previous example for the definition) then A is a compact endomorphism of $L^2_{Hol}(\Omega)$ and $A \in E(a, 1/d)$, where a depends on the geometry of Ω and $\cup_k \phi_k(\Omega)$ (see [BanJ1]).

Operators of this type, known as *transfer operators*, play an important role in the ergodic theory of expanding dynamical systems due to the remarkable fact that their spectral data can be used to gain insight into geometric and dynamic invariants of a given expanding dynamical system (see [Rue]). As a consequence, it is of interest to determine spectral properties of these operators exactly, or at least to a given accuracy. The latter problem, namely that of calculating rigorous error bounds for spectral approximation procedures for these operators provided one of the main motivations to study operators in exponential classes (see [BanJ2]).

We shall now study some of the properties of the classes $E(a, \alpha)$. First we note that if we order the indices (a, α) reverse lexicographically, that is, by defining

$$(a, \alpha) \prec (a', \alpha') :\Leftrightarrow (\alpha < \alpha') \text{ or } (\alpha = \alpha' \text{ and } a < a'),$$

then we obtain the following inclusions.

Proposition 2.4. *Let $a, a' > 0$ and $\alpha, \alpha' > 0$. Then*

(i) $(a, \alpha) \prec (a', \alpha') \Leftrightarrow \mathcal{E}(a', \alpha') \subsetneqq \mathcal{E}(a, \alpha)$;

(ii) $(a, \alpha) \prec (a', \alpha') \Leftrightarrow E(a', \alpha') \subsetneqq E(a, \alpha)$.

Proof. The proof of (i) is straightforward and will be omitted. Assertion (ii) follows from (i) together with the observation that $A \in E(a, \alpha)$ iff $s(A) \in \mathcal{E}(a, \alpha)$ and the fact that for every monotonically decreasing $x \in \mathcal{E}(a, \alpha)$ there is a compact A with $s(A) = x$.

□

While $E(a, \alpha)$ is not, in general, a linear space, it does enjoy the following closure properties.

Proposition 2.5. *Let $a, \alpha > 0$. If $A \in L(H_2, H_1)$, $B \in E(a, \alpha; H_3, H_2)$, and $C \in L(H_4, H_3)$, then $|ABC|_{a,\alpha} \leq \|A\| \, |B|_{a,\alpha} \, \|C\|$. In particular,*

$$L(H_2, H_1)\, E(a, \alpha; H_3, H_2)\, L(H_4, H_2) \subset E(a, \alpha; H_4, H_1).$$

Proof. Follows from

$$s_k(ABC) \leq \|A\| \, s_k(B) \, \|C\|$$

for $k \in \mathbb{N}$ (see [Pie, 2.2]). □

Remark 2.6. The proposition implies that

$$L(H, H)\, E(a, \alpha; H, H)\, L(H, H) \subset E(a, \alpha; H, H)\,.$$

Thus the classes $E(a, \alpha; H, H)$, while lacking linear structure, satisfy part of the definition of an operator ideal. In other words, $E(a, \alpha; H, H)$ is what is sometimes referred to as a pre-ideal (see, for example, [Nel]).

We now consider in more detail the relation between different exponential classes under addition. We start with a general result concerning the singular numbers of a sum of operators.

Proposition 2.7. *Let $A_k \in S_\infty(H_1, H_2)$ for $1 \leq k \leq K$. Then*

$$s_n\left(\sum_{k=1}^{K} A_k\right) \leq K\sigma_n \quad (n \in \mathbb{N}),$$

where σ denotes the decreasing arrangement (see the Appendix) of the K singular number sequences $s(A_1), \ldots, s(A_K)$.

Proof. Set $A := \sum_{k=1}^{K} A_k$. The compactness of the A_k means they have Schmidt representations

$$A_k = \sum_{l=1}^{\infty} s_l(A_k) a_l^{(k)} \otimes b_l^{(k)},$$

where $\{a_l^{(k)}\}_{l\in\mathbb{N}}$ and $\{b_l^{(k)}\}_{l\in\mathbb{N}}$ are suitable orthonormal systems in H_1 and H_2 respectively. Here $a \otimes b$, where $a \in H_1$ and $b \in H_2$, denotes the rank-1 operator $H_1 \to H_2$ given by

$$(a \otimes b)x = (x, a)_{H_1}\, b\,.$$

Let $\nu : \mathbb{N} \to \mathbb{N}$ and $\kappa : \mathbb{N} \to \mathbb{N}$ be functions that effect the decreasing arrangement of the singular numbers of the A_k in the sense that

$$\sigma_n = s_{\nu(n)}(A_{\kappa(n)}) \quad (n \in \mathbb{N}).$$

Then

$$A = \sum_{n=1}^{\infty} \sigma_n a_{\nu(n)}^{(\kappa(n))} \otimes b_{\nu(n)}^{(\kappa(n))},$$

which suggests defining, for each $m \in \mathbb{N}_0$, the rank-m operator $F_m : H_1 \to H_2$ by

$$\begin{aligned} F_0 &:= 0, \\ F_m &:= \sum_{n=1}^{m} \sigma_n a_{\nu(n)}^{(\kappa(n))} \otimes b_{\nu(n)}^{(\kappa(n))} \quad (m \in \mathbb{N}). \end{aligned}$$

If $x \in H_1$ and $y \in H_2$ then

$$\begin{aligned} |((A - F_{m-1})x, y)_{H_2}| &\leq \sum_{n=m}^{\infty} \sigma_n \left| (x, a_{\nu(n)}^{(\kappa(n))})_{H_1} (b_{\nu(n)}^{(\kappa(n))}, y)_{H_2} \right| \\ &\leq \sigma_m \sum_{n=1}^{\infty} \left| (x, a_{\nu(n)}^{(\kappa(n))})_{H_1} (b_{\nu(n)}^{(\kappa(n))}, y)_{H_2} \right| \\ &= \sigma_m \sum_{k=1}^{K} \sum_{l=1}^{\infty} \left| (x, a_l^{(k)})_{H_1} (b_l^{(k)}, y)_{H_2} \right| \\ &\leq \sigma_m \sum_{k=1}^{K} \sqrt{\sum_{l=1}^{\infty} \left| (x, a_l^{(k)})_{H_1} \right|^2} \sqrt{\sum_{l=1}^{\infty} \left| (b_l^{(k)}, y)_{H_2} \right|^2} \\ &\leq \sigma_m \sum_{k=1}^{K} \|x\|_{H_1} \|y\|_{H_2} \\ &= \sigma_m K \|x\|_{H_1} \|y\|_{H_2}. \end{aligned}$$

This estimate justifies the rearrangements (since the series are absolutely convergent) and also yields $\|A - F_{m-1}\| \leq K\sigma_m$, from which the assertion follows. □

Proposition 2.8. *Suppose that* $A_n \in E(a_n, \alpha; H_1, H_2)$ *for* $1 \leq n \leq K$. *Let* $A := \sum_{n=1}^{K} A_n$ *and* $a' := (\sum_{n=1}^{K} a_n^{-1/\alpha})^{-\alpha}$. *Then*

(i) $A \in E(a', \alpha)$ *with* $|A|_{a',\alpha} \leq K \max_{1 \leq n \leq K} |A_n|_{a_n,\alpha}$. *In particular*

$$E(a_1, \alpha) + \cdots + E(a_K, \alpha) \subset E(a', \alpha).$$

(ii) *If both* H_1 *and* H_2 *are infinite-dimensional then the inclusion above is sharp in the sense that*

$$E(a_1, \alpha) + \cdots + E(a_K, \alpha) \not\subset E(b, \alpha),$$

whenever $b > a'$.

Proof. Assertion (i) follows from Proposition 2.7 and Corollary 5.4 (i), which gives an upper bound on the rate of decay of the decreasing arrangement of the sequences $s(A_1), \ldots, s(A_K)$.

For the proof of (ii) define K sequences $s^{(1)},\dots,s^{(K)}$ by

$$s_n^{(k)} := \exp(-a_k n^\alpha) \quad (n\in\mathbb{N}).$$

It turns out that it suffices to exhibit K compact operators A_k with $s_n(A_k)=s_n^{(k)}$ such that $s(A)$ is the decreasing arrangement of the sequences $s^{(1)},\dots,s^{(K)}$. To see this, note that then $A_k\in E(a_k,\alpha)$ for $k\in\{1,\cdots,K\}$. At the same time $s_n(A)\geq \exp(-a'(n+K)^\alpha)$ by Corollary 5.4 (ii), so that $A\notin E(b,\alpha)$ whenever $b>a'$.

In order to construct these operators we proceed as follows. Since each H_i was assumed to be infinite-dimensional we can choose an orthonormal basis $\{h_n^{(i)}\}_{n\in\mathbb{N}}$ for each of them. For each $k=1,\dots,K$, we now define a compact operator $A_k: H_1\to H_2$ by

$$A_k h_n^{(1)} := \begin{cases} s^{(k)}_{(n+(k-1))/K} h_n^{(2)} & \text{for } n\in K\mathbb{N}-(k-1),\\ 0 & \text{for } n\notin K\mathbb{N}-(k-1).\end{cases}$$

It is not difficult to see that $s_n(A_k)=s_n^{(k)}$. Moreover, it is easily verified that the singular numbers of A are precisely the numbers of the form $s_n^{(k)}$ with $n\in\mathbb{N}$ and $k=1,\dots,K$. Thus, $s(A)$ is the decreasing arrangement of the sequences $s^{(1)},\dots,s^{(K)}$ as required.

□

Remark 2.9. The proposition implies that $E(a,\alpha)+E(a,\alpha)\subset E(2^{-\alpha}a,\alpha)$, but $E(a,\alpha)+E(a,\alpha)\not\subset E(a,\alpha)$, because $2^{-\alpha}a<a$. In particular, $E(a,\alpha)$ is not a linear space.

The following result establishes a sharp bound on the eigenvalue decay rate in each exponential class.

Proposition 2.10. *Let $a,\alpha>0$ and $A\in E(a,\alpha;H,H)$. Then*

$$\lambda(A)\in\mathcal{E}(a/(1+\alpha),\alpha) \text{ with } |\lambda(A)|_{a/(1+\alpha),\alpha}\leq |A|_{a,\alpha}.$$

If H is infinite-dimensional, the result is sharp in the sense that there is an operator $A\in E(a,\alpha;H,H)$ such that $\lambda(A)\notin\mathcal{E}(b,\alpha)$ whenever $b>a/(1+\alpha)$.

Proof. If $A\in E(a,\alpha)$ then $s_k(A)\leq |A|_{a,\alpha}\exp(-ak^\alpha)$. Using the multiplicative Weyl inequality [Pie, 3.5.1] we have

$$|\lambda_k(A)|^k \leq \prod_{l=1}^k |\lambda_l(A)| \leq \prod_{l=1}^k s_l(A) \leq \prod_{l=1}^k |A|_{a,\alpha}\exp(-al^\alpha) = \\ = |A|_{a,\alpha}^k \exp(-a\sum_{l=1}^k l^\alpha). \quad (1)$$

But $\sum_{l=1}^k l^\alpha \geq \int_0^k x^\alpha\,dx = \frac{1}{1+\alpha}k^{\alpha+1}$, which combined with (1) yields

$$|\lambda_k(A)|\leq |A|_{a,\alpha}\exp(-ak^\alpha/(1+\alpha)).$$

Sharpness is proved in several steps. We start with the following observation. Let $\tau_1 \geq \ldots \geq \tau_N \geq 0$ be positive real numbers. Consider the matrix $C(\tau_1, \ldots, \tau_N) \in L(\mathbb{C}^N, \mathbb{C}^N)$ given by

$$C(\tau_1, \ldots, \tau_N) := \begin{pmatrix} 0 & \tau_1 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & \tau_{N-1} \\ \tau_N & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

It easy to see that

$$s_n(C(\tau_1, \ldots, \tau_N)) = \tau_n$$

and

$$|\lambda_1(C(\tau_1, \ldots, \tau_N))| = \cdots = |\lambda_N(C(\tau_1, \ldots, \tau_N))| = (\tau_1 \cdots \tau_N)^{1/N}.$$

The desired operator is constructed as follows. Fix $a > 0$ and $\alpha > 0$. Next choose a super-exponentially increasing sequence N_n, that is, N_n is increasing and $\lim_{n\to\infty} N_{n-1}/N_n = 0$. For definiteness we could set $N_n = \exp(n^2)$.

Put $N_0 = 0$ and define

$$d_n := N_n - N_{n-1} \quad (n \in \mathbb{N}).$$

Define matrices $A_n \in L(\mathbb{C}^{d_n}, \mathbb{C}^{d_n})$ by

$$A_n = C(\exp(-a(N_{n-1}+1)^\alpha), \ldots, \exp(-a(N_n)^\alpha)).$$

Then

$$s_k(A_n) = \exp(-a(N_{n-1}+k)^\alpha) \quad (1 \leq k \leq d_n)$$

and

$$|\lambda_k(A_n)| = \exp(-ap_n^\alpha) \quad (1 \leq k \leq d_n),$$

where

$$p_n := \frac{1}{d_n} \sum_{l=N_{n-1}+1}^{N_n} l^\alpha.$$

Put

$$H := \bigoplus_{n=1}^{\infty} \mathbb{C}^{d_n},$$

and let $A : H \to H$ be the block-diagonal operator

$$(Ax)_n = A_n x_n.$$

Clearly, the singular numbers of A are given by $s_k(A) = \exp(-ak^\alpha)$ and the moduli of the eigenvalues are the numbers $\exp(-ap_n^\alpha)$ occurring with multiplicity d_n.

Before checking that A has the desired properties we observe that

$$p_n^\alpha = \frac{1}{d_n} \sum_{l=N_{n-1}+1}^{N_n} l^\alpha \leq \frac{1}{d_n} \int_{N_{n-1}+1}^{N_n+1} x^\alpha =$$

$$= \frac{1}{\alpha+1} \frac{1}{d_n} ((N_n+1)^{\alpha+1} - (N_{n-1}+1)^{\alpha+1}) = \frac{1}{\alpha+1} N_n^\alpha \delta_n, \quad (2)$$

with $\lim_{n\to\infty} \delta_n = 1$. The latter follows from the fact that the sequence N_n was chosen to be super-exponentially increasing.

Suppose now that $b > a/(\alpha+1)$. Since $|\lambda_{N_n}(A)| = \exp(-ap_n^\alpha)$ we have

$$|\lambda_{N_n}(A)| \exp(bN_n^\alpha) \geq \exp(-\frac{a}{\alpha+1} N_n^\alpha \delta_n + bN_n^\alpha) = \exp(N_n^\alpha (b - \frac{a}{\alpha+1}\delta_n)).$$

Thus

$$|\lambda_{N_n}(A)| \exp(bN_n^\alpha) \to +\infty \text{ as } n \to \infty,$$

which means that $\lambda(A) \notin \mathcal{E}(b,\alpha)$. □

Remark 2.11. Similar results have been obtained by König and Richter [KR, Proposition 1], though without estimates on the gauge of $\lambda(A)$.

3. Resolvent estimates

In this section we shall derive an upper bound for the norm of the resolvent $(zI-A)^{-1}$ of $A \in E(a,\alpha)$ in terms of the distance of z to the spectrum of A and the departure from normality of A, a number quantifying the non-normality of A. We shall employ a technique originally due to Henrici [Hen], who used it in a finite-dimensional context. The basic idea is to write A as a perturbation of a normal operator having the same spectrum as A by a quasi-nilpotent operator. A similar argument can be used to derive resolvent estimates for operators belonging to Schatten classes (see [Gil] (and references therein) and [Ban]).

Following the idea outlined above we start with bounds for quasi-nilpotent operators.

Proposition 3.1. *Let $a, \alpha > 0$.*

(i) *If $A \in E(a,\alpha; H, H)$ is quasi-nilpotent, that is, $\sigma(A) = \{0\}$, then*

$$\left\|(I-A)^{-1}\right\| \leq f_{a,\alpha}(|A|_{a,\alpha}), \tag{3}$$

where $f_{a,\alpha} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is defined by

$$f_{a,\alpha}(r) = \prod_{n=1}^\infty (1 + r\exp(-an^\alpha)).$$

Moreover, $f_{a,\alpha}$ has the following asymptoticss:

$$\log f_{a,\alpha}(r) \sim a^{-1/\alpha} \frac{\alpha}{1+\alpha} (\log r)^{1+1/\alpha} \text{ as } r \to \infty. \tag{4}$$

(ii) *If H is infinite-dimensional the estimate (3) is sharp in the sense that there is a quasi-nilpotent $B \in E(a,\alpha; H, H)$ such that*

$$\log \left\|(I - zB)^{-1}\right\| \sim \log f_{a,\alpha}(|zB|_{a,\alpha}) \text{ as } |z| \to \infty. \tag{5}$$

Proof. Fix $a, \alpha > 0$.

(i) Since A is trace class and quasi-nilpotent a standard estimate (see, for example, [GGK, Chapter X, Theorem 1.1]) shows that

$$\left\|(I-A)^{-1}\right\| \leq \prod_{n=1}^\infty (1 + s_n(A)).$$

Thus

$$\left\|(I-A)^{-1}\right\| \leq \prod_{n=1}^\infty (1 + |A|_{a,\alpha} \exp(-an^\alpha)) = f_{a,\alpha}(|A|_{a,\alpha}).$$

It remains to prove the growth estimate (4). We proceed by noting that $f_{a,\alpha}$ extends to an entire function of genus zero with $f_{a,\alpha}(0) = 1$. Moreover, the maximum modulus of $f_{a,\alpha}(z)$ for $|z| = r$ equals $f_{a,\alpha}(r)$. The growth of $f_{a,\alpha}$ can thus be estimated by

$$N(r) \leq \log f_{a,\alpha}(r) \leq N(r) + Q(r)\,, \tag{6}$$

where $N(r) = \int_0^r t^{-1}n(t)\,dt$, $Q(r) = r\int_r^\infty t^{-2}n(t)\,dt$ and $n(r)$ denotes the number of zeros of $f_{a,\alpha}$ lying in the closed disk with radius r centred at 0 (see [Boa, p. 47]).

Since $n(r) = \lfloor a^{-1/\alpha}(\log_+ r)^{1/\alpha}\rfloor$, where $\log_+(r) = \max\{0, \log r\}$ and $\lfloor\cdot\rfloor$ denotes the floor-function, we have, for $r \geq 1$,

$$\begin{aligned} N(r) &= a^{-1/\alpha}\int_1^r t^{-1}(\log t)^{1/\alpha}\,dt + O(\log r) \\ &= a^{-1/\alpha}\frac{\alpha}{1+\alpha}(\log r)^{1+1/\alpha} + O(\log r)\,; \end{aligned} \tag{7}$$

while Q satisfies

$$Q(r) = O((\log r)^{1/\alpha}) \text{ as } r \to \infty\,. \tag{8}$$

To see this, note that

$$Q(r) \leq a^{-1/\alpha} r\int_r^\infty t^{-2}(\log t)^{1/\alpha}\,dt = a^{-1/\alpha} r\int_{\log r}^\infty e^{-u}u^{1/\alpha}\,du\,; \tag{9}$$

putting $r = e^s$ it thus suffices to show that

$$e^s\int_s^\infty e^{-u}u^{1/\alpha}\,du = O(s^{1/\alpha}) \text{ as } s \to \infty.$$

This, however, is the case since

$$s^{-1/\alpha}e^s\int_s^\infty e^{-u}u^{1/\alpha}\,du = \int_s^\infty e^{-(u-s)}(u/s)^{1/\alpha}\,du = \int_0^\infty e^{-t}(1+t/s)^{1/\alpha}\,dt \to 1$$

as $s \to \infty$. Combining (8), (7) and (6) the growth estimate (4) follows.

(ii) Since H is infinite-dimensional, we may choose an orthonormal basis $\{h_n\}_{n\in\mathbb{N}}$. Define the operator $B \in L(H,H)$ by

$$Bh_n := \exp(-an^\alpha)h_{n+1} \quad (n \in \mathbb{N}).$$

It is not difficult to see that $s_n(B) = \exp(-an^\alpha)$ for $n \in \mathbb{N}$, so that $B \in E(a,\alpha; H, H)$. Before we proceed let

$$c_n := \sum_{k=1}^n k^\alpha \quad (n \in \mathbb{N}_0),$$

and note that since $\int_0^n x^\alpha\,dx \leq \sum_{k=1}^n k^\alpha \leq \int_1^{n+1} x^\alpha\,dx$, we have

$$\frac{1}{\alpha+1}n^{\alpha+1} \leq c_n \leq \frac{1}{\alpha+1}(n+1)^{\alpha+1} \quad (n \in \mathbb{N}_0).$$

The operator B is quasi-nilpotent, since

$$\|B^n\| = \exp(-ac_n) \leq \exp(-\frac{a}{\alpha+1}n^{\alpha+1})\,,$$

which implies $\|B^n\|^{1/n} \to 0$ as $n \to \infty$.

It order to determine the asymptotics of $\log\|(I-zB)^{-1}\|$ we start by noting that

$$\left\|(I-zB)^{-1}\right\|^2 \geq \left\|(I-zB)^{-1}h_1\right\|^2 = \|\sum_{n=0}^{\infty}(zB)^n h_1\|^2$$
$$= \sum_{n=0}^{\infty}|z|^{2n}\exp(-2ac_n) \geq \sum_{n=0}^{\infty}|z|^{2n}\exp(-2\frac{a}{\alpha+1}(n+1)^{\alpha+1}) \geq |z|^{-2}g(|z|), \quad (10)$$

where

$$g(r) := \sum_{n=1}^{\infty} r^{2n}\exp(-2\frac{a}{\alpha+1}n^{\alpha+1}) \quad (r\in\mathbb{R}_0^+).$$

Thus

$$2\log f_{a,\alpha}(|zB|_{a,\alpha}) \geq 2\log\left\|(I-zB)^{-1}\right\| \geq -2\log|z| + \log g(|z|)\,, \quad (11)$$

which shows that in order to obtain the desired asymptotics (5) it suffices to prove that

$$\log g(r) \sim 2a^{-1/\alpha}\frac{\alpha}{\alpha+1}(\log r)^{1+1/\alpha} \text{ as } r\to\infty. \quad (12)$$

In order to establish the asymptotics above we introduce the maximum term

$$\mu(r) := \max_{1\leq n<\infty} r^{2n}\exp(-2\frac{a}{\alpha+1}n^{\alpha+1}) \quad (r\in\mathbb{R}_0^+)\,. \quad (13)$$

Since g extends to an entire function of finite order we have (see, for example, [PS, Problem 54])

$$\log\mu(r) \sim \log g(r) \text{ as } r\to\infty\,,$$

which implies that it now suffices to show that μ has the desired asymptotics

$$\log\mu(r) \sim 2a^{-1/\alpha}\frac{\alpha}{\alpha+1}(\log r)^{1+1/\alpha} \text{ as } r\to\infty. \quad (14)$$

We now estimate $\mu(r)$ for fixed r. Define the function $m_r:\mathbb{R}_0^+\to\mathbb{R}_0^+$ by

$$m_r(x) = \exp(-2\frac{a}{\alpha+1}x^{\alpha+1} + 2x\log r).$$

It turns out that m_r has a maximum at $x_r = a^{-1/\alpha}(\log r)^{1/\alpha}$ and that m_r is monotonically increasing on $(0,x_r)$ and monotonically decreasing on (x_r,∞). Thus

$$\log m_r(x_r-1) \leq \log\mu(r) \leq \log m_r(x_r)\,. \quad (15)$$

Write $x_r - 1 = \delta_r x_r$ and note that $\delta_r\to 1$ as $r\to\infty$. Observing that

$$\frac{\log m_r(x_r)}{(\log r)^{1+1/\alpha}} = 2a^{-1/\alpha}\frac{\alpha}{\alpha+1}$$

while

$$\frac{\log m_r(x_r-1)}{(\log r)^{1+1/\alpha}} = \frac{\log m_r(\delta_r x_r)}{(\log r)^{1+1/\alpha}} = 2a^{-1/\alpha}\left(-\frac{\delta_r^{\alpha+1}}{\alpha+1}+\delta_r\right) \to 2a^{-1/\alpha}\frac{\alpha}{\alpha+1}$$

as $r\to\infty$ we conclude, using (15), that (14) holds. This implies (12), which yields (5), as required. □

Remark 3.2.
(i) The bound for the growth of the resolvent of a quasi-nilpotent $A \in E(a, \alpha)$ given in the above proposition is an improvement compared to those obtainable from the usual estimates for operators in the Schatten classes. Indeed, if A belongs to the Schatten p-class (i.e. $s(A)$ is p-summable) for some $p > 0$, then

$$\left\|(I - A)^{-1}\right\| \leq f_p(\|A\|_p)$$

where $\|A\|_p$ denotes the Schatten p-(quasi) norm of A. Here, $f_p : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is given by

$$f_p(r) = \exp(a_p r^p + b_p)\,,$$

where a_p and b_p are positive numbers depending on p, but not on A (see, for example, [Sim1] or [Ban, Theorem 2.1], where a discussion of the constants a_p and b_p can be found).

(ii) Closer inspection of the proof yields the following explicit upper bound for $f_{a,\alpha}$

$$\log f_{a,\alpha}(r) \leq a^{-1/\alpha} \left(\frac{\alpha}{1+\alpha} (\log_+ r)^{1+1/\alpha} + r\Gamma(1 + 1/\alpha, \log_+ r) \right)\,,$$

where $\Gamma(\beta, s)$ denotes the incomplete gamma function

$$\Gamma(\beta, s) = \int_s^\infty \exp(-t)\, t^{\beta-1}\, dt\,.$$

This follows from (6) together with the estimate $n(r) \leq a^{-1/\alpha}(\log_+ r)^{1/\alpha}$.

A simple consequence of the proposition is the following estimate for the growth of the resolvent of a quasi-nilpotent $A \in E(a, \alpha)$.

Corollary 3.3. *If $A \in E(a, \alpha)$ is quasi-nilpotent, then*

$$\left\|(zI - A)^{-1}\right\| \leq |z|^{-1} f_{a,\alpha}(|z|^{-1}|A|_{a,\alpha}) \text{ for } z \neq 0.$$

The proposition above can be used to obtain growth estimates for the resolvents of any $A \in E(a, \alpha)$ by means of the following device.

Theorem 3.4. *Let $A \in S_\infty$. Then A can be written as a sum*

$$A = D + N,$$

such that

(i) *$D \in S_\infty$, $N \in S_\infty$;*
(ii) *D is normal and $\lambda(D) = \lambda(A)$;*
(iii) *N and $(zI - D)^{-1}N$ are quasi-nilpotent for every $z \in \varrho(D) = \varrho(A)$.*

Proof. See [Ban, Theorem 3.2]. □

This result motivates the following definition.

Definition 3.5. Let $A \in S_\infty$. A decomposition

$$A = D + N$$

with D and N enjoying properties (i–iii) of the previous theorem is called a *Schur decomposition of* A. The operators D and N will be referred to as the *normal* and the *quasi-nilpotent part of the Schur decomposition of* A, respectively.

Remark 3.6.

(i) The terminology stems from the fact that in the finite dimensional setting the decomposition in Theorem 3.4 can be obtained as follows: since any matrix is unitarily equivalent to an upper-triangular matrix by a classical result due to Schur, it suffices to establish the result for matrices of this form. In this case, simply choose D to be the diagonal part, and N the off-diagonal part of the matrix.

(ii) The decomposition is not unique, not even modulo unitary equivalence: there is a matrix A with two Schur decompositions $A = D_1 + N_1$ and $A = D_2 + N_2$ such that N_1 is not unitarily equivalent to N_2 (see [Ban, Remark 3.5 (i)]). Note, however, that the normal parts of any two Schur decompositions of a given compact operator are always unitarily equivalent.

Using the results in the previous section we are able to locate the position of the normal part and the quasi-nilpotent part of an operator $A \in E(a, \alpha)$ in the scale of exponential classes.

Proposition 3.7. *Let $A \in E(a, \alpha)$. If $A = D + N$ is a Schur decomposition of A with normal part D and quasi-nilpotent part N, then*

(i) $D \in E(a/(1+\alpha), \alpha)$ *with* $|D|_{a/(1+\alpha),\alpha} \leq |A|_{a,\alpha}$*;*

(ii) $N \in E(a', \alpha)$ *with* $|N|_{a',\alpha} \leq 2|A|_{a,\alpha}$*, where* $a' = a(1 + (1+\alpha)^{1/\alpha})^{-\alpha}$*.*

Proof. Since D is normal, its singular numbers coincide with its eigenvalues, which in turn coincide with the eigenvalues of A. Assertion (i) is thus a consequence of Proposition 2.10, while assertion (ii) follows from (i) and Proposition 2.8 by taking $N = A - D$. □

Remark 3.8. Assertion (i) above is sharp in the following sense: there is $A \in E(a, \alpha)$ such that for any normal part D of A we have $D \notin E(b, \alpha)$ whenever $b > a/(1 + \alpha)$. This follows from the corresponding statement in Proposition 2.10 and the fact that all normal parts of A are unitarily equivalent.

For later use we define the following quantities, originally introduced by Henrici [Hen].

Definition 3.9. Let $a, \alpha > 0$ and define $\nu_{a,\alpha} : S_\infty \to \mathbb{R}_0^+ \cup \{\infty\}$ by

$$\nu_{a,\alpha}(A) := \inf \{ |N|_{a,\alpha} \, : \, N \text{ is a quasi-nilpotent part of } A \} \, .$$

We call $\nu_{a,\alpha}(A)$ the (a, α)-*departure from normality of A.*

Remark 3.10. Henrici originally introduced this quantity for matrices and with the (a, α)-gauge of N replaced by the Hilbert-Schmidt norm. For a discussion of the case where the (a, α)-gauge is replaced by a Schatten norm and its uses to obtain resolvent estimates for Schatten class operators see [Ban].

The term 'departure from normality' is justified in view of the following characterisation.

Proposition 3.11. *Let $A \in E(a, \alpha)$. Then*

$$\nu_{a,\alpha}(A) = 0 \Leftrightarrow A \text{ is normal.}$$

Proof. Let $\nu_{a,\alpha}(A) = 0$. Then there exists a sequence of Schur decompositions with quasi-nilpotent parts N_n such that $|N_n|_{a,\alpha} \to 0$. Thus

$$\|A - D_n\| = \|N_n\| = s_1(N_n) \le \exp(-a)|N|_{a,\alpha} \to 0,$$

where D_n are the corresponding normal parts. Thus A is a limit of normal operators converging in the uniform operator topology and is therefore normal. The converse is trivial. □

For a given $A \in E(a, \alpha)$ the departure from normality is difficult to calculate. The following simple but somewhat crude bound is useful in practice.

Proposition 3.12. *Let $A \in E(a, \alpha)$. Then*

$$\nu_{b,\alpha}(A) \le 2|A|_{a,\alpha}$$

whenever $b \le a(1 + (1 + \alpha)^{1/\alpha})^{-\alpha}$.

Proof. Follows from Proposition 3.7 together with the fact that $|N|_{b,\alpha} \le |N|_{a',\alpha}$ whenever $b \le a'$. □

We are now ready to deduce resolvent estimates for arbitrary $A \in E(a, \alpha)$. Using a Schur decomposition $A = D + N$ with D normal and N quasi-nilpotent we consider A as a perturbation of D by N. Since D is normal we have

$$\left\|(zI - D)^{-1}\right\| = \frac{1}{d(z, \sigma(D))} \quad (z \in \varrho(D)), \tag{16}$$

where for $z \in \mathbb{C}$ and $\sigma \subset \mathbb{C}$ closed,

$$d(z, \sigma) := \inf_{\lambda \in \sigma} |z - \lambda|$$

denotes the distance of z to σ. The influence of the perturbation N on the other hand, is controlled by Proposition 3.1. All in all, we have the following.

Theorem 3.13. *Let $A \in E(a, \alpha)$. If $b \le a(1 + (1 + \alpha)^{1/\alpha})^{-\alpha}$, then*

$$\left\|(zI - A)^{-1}\right\| \le \frac{1}{d(z, \sigma(A))} f_{b,\alpha}\left(\frac{\nu_{b,\alpha}(A)}{d(z, \sigma(A))}\right). \tag{17}$$

Proof. Fix $b \le a(1+(1+\alpha)^{1/\alpha})^{-\alpha}$. Then there is a Schur decomposition of A with normal part D and quasi-nilpotent part $N \in E(b, \alpha)$ by Proposition 3.7. Since the bound above is trivial for $z \in \sigma(A)$ we may assume $z \in \varrho(A)$. As D and N stem from a Schur decomposition of A we see that $(zI - D)^{-1}$ exists (because $\sigma(A) = \sigma(D)$) and that $(zI - D)^{-1}N$ is quasi-nilpotent. Moreover

$$|(zI - D)^{-1}N|_{b,\alpha} \le \left\|(zI - D)^{-1}\right\| |N|_{b,\alpha} = \frac{|N|_{b,\alpha}}{d(z, \sigma(D))},$$

by (16) and Proposition 2.5. Thus $(I - (zI - D)^{-1}N)$ is invertible in L and

$$\left\|(I - (zI - D)^{-1}N)^{-1}\right\| \le f_{b,\alpha}\left(\frac{|N|_{b,\alpha}}{d(z, \sigma(D))}\right),$$

by Proposition 3.1. Now, since $(zI-A) = (zI-D)(I-(zI-D)^{-1}N)$, we conclude that $(zI-A)$ is invertible in L and

$$\begin{aligned} \left\|(z-A)^{-1}\right\| &\leq \left\|(I-(zI-D)^{-1}N)^{-1}\right\| \left\|(zI-D)^{-1}\right\| \\ &\leq \frac{1}{d(z,\sigma(D))} f_{b,\alpha}\left(\frac{|N|_{b,\alpha}}{d(z,\sigma(D))}\right). \end{aligned}$$

Taking the infimum over all Schur decompositions while using $\sigma(A) = \sigma(D)$ once again the result follows. ☐

Remark 3.14.
(i) The estimate remains valid if $\nu_{b,\alpha}(A)$ is replaced by something larger, for example by the upper bound given in Proposition 3.12.
(ii) The estimate is sharp in the sense that if A is normal then (17) reduces to the sharp estimate (16).

4. Bounds for the spectral distance

Using the resolvent estimates obtained in the previous section it is possible to give upper bounds for the Hausdorff distance of the spectra of operators in $E(a,\alpha)$. Recall that the *Hausdorff distance* $\text{Hdist}\,(.,.)$ is the following metric defined on the space of compact subsets of $\mathbb{C}$

$$\text{Hdist}\,(\sigma_1,\sigma_2) := \max\{\hat{d}(\sigma_1,\sigma_2), \hat{d}(\sigma_2,\sigma_1)\},$$

where

$$\hat{d}(\sigma_1,\sigma_2) := \sup_{\lambda\in\sigma_1} d(\lambda,\sigma_2)$$

and σ_1 and σ_2 are two compact subsets of $\mathbb{C}$.

For $A, B \in L$ we borrow terminology from matrix perturbation theory and call $\hat{d}(\sigma(A),\sigma(B))$ the *spectral variation of A with respect to B* and $\text{Hdist}\,(\sigma(\text{A}),\sigma(\text{B}))$ the *spectral distance of A and B*.

The main tool to bound the spectral variation is the following result, which is based on a simple but powerful argument usually credited to Bauer and Fike [BauF] who first employed it in a finite-dimensional context.

Proposition 4.1. *Let $A \in S_\infty$. Suppose that there is a strictly monotonically increasing surjection $g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that*

$$\left\|(zI-A)^{-1}\right\| \leq g(d(z,\sigma(A))^{-1}) \quad (\forall z \in \varrho(A)).$$

Then for any $B \in L$, the spectral variation of B with respect to A satisfies

$$\hat{d}(\sigma(B),\sigma(A)) \leq h(\|A-B\|),$$

where $h : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is the function defined by

$$h(r) = (\tilde{g}(r^{-1}))^{-1}$$

and $\tilde{g} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is the inverse of the function g.

Proof. Let $B \in L$. In what follows we shall use the abbreviations

$$d := d(z,\sigma(A)), \quad E := B - A.$$

Without loss of generality we may assume that $E \neq 0$. We shall first establish the following implication:

$$z \in \sigma(B) \cap \varrho(A) \Rightarrow \|E\|^{-1} \leq \left\|(zI - A)^{-1}\right\| . \tag{18}$$

To see this let $z \in \sigma(B) \cap \varrho(A)$ and suppose to the contrary that

$$\left\|(zI - A)^{-1}\right\| \|E\| < 1.$$

Then $\big(I - (zI - A)^{-1}E\big)$ is invertible in L, which implies that $(zI - B) = (zI - A)\big(I - (zI - A)^{-1}E\big)$ is invertible in L. Thus $z \in \varrho(B)$ which contradicts $z \in \sigma(B)$. Thus the implication (18) holds.

In order to prove the proposition it suffices to show that

$$z \in \sigma(B) \Rightarrow d(z, \sigma(A)) \leq h(\|E\|) , \tag{19}$$

which is proved as follows. Let $z \in \sigma(B)$. If $z \in \sigma(A)$ there is nothing to prove. We may thus assume that $z \in \varrho(A)$. Hence, by (18),

$$\|E\|^{-1} \leq \left\|(zI - A)^{-1}\right\| \leq g(d^{-1}).$$

Since g is strictly monotonically increasing, so is $\tilde{g}$. Thus

$$\tilde{g}(\|E\|^{-1}) \leq d^{-1},$$

and hence

$$d(z, \sigma(A)) = d \leq (\tilde{g}(\|E\|^{-1}))^{-1} = h(\|E\|) = h(\|A - B\|).$$

□

Using the proposition above together with the resolvent estimates in Theorem 3.13 we now obtain the following spectral variation and spectral distance formulae.

Theorem 4.2. *Let $A \in E(a, \alpha)$ and define $a' := a(1 + (1 + \alpha)^{1/\alpha})^{-\alpha}$.*

(i) *If $B \in L$ and $b > a'$ then*

$$\hat{d}(\sigma(B), \sigma(A)) \leq \nu(A)_{b,\alpha} h_{b,\alpha} \left(\frac{\|A - B\|}{\nu_{b,\alpha}(A)} \right) . \tag{20}$$

(ii) *If $B \in E(a, \alpha)$ and $b > a'$ then*

$$\mathrm{Hdist}\,(\sigma(A), \sigma(B)) \leq m h_{b,\alpha} \left(\frac{\|A - B\|}{m} \right) , \tag{21}$$

where $m := \max\{\nu(A)_{b,\alpha}, \nu(B)_{b,\alpha}\}$.

Here, the function $h_{b,\alpha} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is given by

$$h_{b,\alpha}(r) := (\tilde{g}_{b,\alpha}(r^{-1}))^{-1},$$

where $\tilde{g} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is the inverse of the function $g_{b,\alpha} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined by

$$g_{b,\alpha}(r) := r f_{b,\alpha}(r) ,$$

and $f_{a,\alpha}$ is the function defined in Proposition 3.1.

Proof. To prove (i) fix $b \geq a'$. The assertion now follows from the previous proposition by noting that

$$\left\|(zI - A)^{-1}\right\| \leq \frac{1}{\nu_{b,\alpha}(A)} g_{b,\alpha}\left(\frac{\nu_{b,\alpha}(A)}{d(z,\sigma(A))}\right)$$

by Theorem 3.13. To prove (ii) fix $b \geq a'$. Then it is not difficult see that

$$\left\|(zI - A)^{-1}\right\| \leq \frac{1}{m} g_{b,\alpha}\left(\frac{m}{d(z,\sigma(A))}\right),$$

and

$$\left\|(zI - B)^{-1}\right\| \leq \frac{1}{m} g_{b,\alpha}\left(\frac{m}{d(z,\sigma(B))}\right),$$

and the assertion follows as in the proof of (i). □

Remark 4.3.

(i) Note that $\lim_{r\downarrow 0} h_{b,\alpha}(r) = 0$, so the estimate for the spectral distance becomes small when $\|A - B\|$ is small. In fact, it can be shown that

$$\log h_{b,\alpha}(r) \sim -b^{1/(1+\alpha)}\left(\frac{1+\alpha}{\alpha}\right)^{\alpha/(1+\alpha)} |\log r|^{\alpha/(1+\alpha)} \text{ as } r \downarrow 0.$$

This follows from the asymptotics in Proposition 3.1 together with the fact that if $\log f(r) \sim a(\log r)^{\beta}$, then $\log \tilde{f}(r) \sim a^{-1/\beta}(\log r)^{1/\beta}$ where $\tilde{f}$ is the inverse of f.

(ii) It is not difficult to see, for example by arguing as in the proof of part (ii) of the theorem, that the inequalities (20) and (21) above remain valid if $\nu_{b,\alpha}(A)$ or $\nu_{b,\alpha}(B)$ is replaced by something larger — for example, by the upper bounds given in Proposition 3.12.

(iii) Assertion (ii) of the theorem is sharp in the sense that if both operators are normal, then (ii) reduces to

$$\mathrm{Hdist}\left(\sigma(A), \sigma(B)\right) \leq \|A - B\| \ .$$

5. Appendix: Monotone arrangements

In this appendix we present a number of results concerning sequences and their arrangements used in Section 2.

Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence. Define

$$\|a\|_+ := \sup_{n\in\mathbb{N}} a_n,$$

$$\|a\|_- := \inf_{n\in\mathbb{N}} a_n.$$

For $u : \mathbb{N} \to \mathbb{R}$ call

$$\mathrm{rank}\,(u) := \mathrm{card}\,\{\, n \in \mathbb{N} \,:\, u_n \neq 0 \,\}\,.$$

Definition 5.1. Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence. Let extended real-valued sequences $a^{(+)}$ and $a^{(-)}$ be defined by

$$a^{(+)} : \mathbb{N} \to \mathbb{R} \cup \{-\infty, \infty\}, \quad a^{(+)}_n := \inf\left\{\, \|a - u\|_+ \,:\, \mathrm{rank}\, u < n \,\right\},$$

$$a^{(-)} : \mathbb{N} \to \mathbb{R} \cup \{-\infty, \infty\}, \quad a^{(-)}_n := \sup\left\{\, \|a - u\|_- \,:\, \mathrm{rank}\, u < n \,\right\}.$$

We call $a^{(+)}$ the *decreasing arrangement of* a, and $a^{(-)}$ the *increasing arrangement of* a.

This terminology is justified in view of the fact that $a^{(+)}$ (respectively $a^{(-)}$) is a decreasing (respectively increasing) sequence. Moreover, if a is monotonically decreasing, then $a^{(+)} = a$, and similarly for monotonically increasing sequences.

More generally, we will consider monotone arrangements of collections of sequences by first amalgamating them into one sequence and then regarding the resulting monotone arrangement. A more precise definition is the following.

Definition 5.2. Given K real-valued sequences $\{a_n^{(1)}\}_{n\in\mathbb{N}}, \dots, \{a_n^{(K)}\}_{n\in\mathbb{N}}$, define a new sequence $a : \mathbb{N} \to \mathbb{R}$ by

$$a_{(k-1)K+i} = a_k^{(i)} \quad \text{ for } k \in \mathbb{N} \text{ and } 1 \le i \le K\,.$$

We then call $a^{(+)}$ ($a^{(-)}$) the *decreasing (increasing) arrangement of the* K *sequences* $a^{(1)}, \cdots, a^{(K)}$.

Our application of decreasing arrangements will typically be to singular number sequences, all of which converge to zero at some stretched exponential rate. Technically and notationally it is preferable to work with the logarithms of reciprocals of such sequences, that is, increasing sequences converging to $+\infty$ at some polynomial rate.

The following is the main result of this appendix.

Proposition 5.3. *Let* $\alpha > 0$ *and* $K \in \mathbb{N}$. *Suppose that for each* $k \in \{1, \dots, K\}$ *we are given a real sequence* $a^{(k)}$, *a positive constant* $a_k > 0$, *and a real number* A_k. *Let* $a^{(-)}$ *denote the increasing arrangement of the* K *sequences* $a^{(1)}, \dots, a^{(K)}$, *and define*

$$c = \left(\sum_{k=1}^{K} a_k^{-1/\alpha} \right)^{-\alpha}\,.$$

(i) *If*

$$a_n^{(k)} \ge a_k n^\alpha + A_k \quad (\forall n \in \mathbb{N},\, k \in \{1, \dots, K\})\,,$$

then

$$a_n^{(-)} \ge c n^\alpha + \min\{A_1, \dots, A_K\} \quad (\forall n \in \mathbb{N})\,.$$

(ii) *If*

$$a_n^{(k)} \le a_k n^\alpha + A_k \quad (\forall n \in \mathbb{N},\, k \in \{1, \dots, K\})\,,$$

then

$$a_n^{(-)} \le c(n+K)^\alpha + \max\{A_1, \dots, A_K\} \quad (\forall n \in \mathbb{N})\,.$$

Proof. For $k \in \{1, \dots, K\}$ set $a_0^{(k)} = -\infty$ and, for $r \in \mathbb{R}$, define the counting functions

$$\mu_k(r) := \operatorname{card}\left\{\, n \in \mathbb{N} \,:\, a_n^{(k)} \le r \,\right\}$$
$$\mu(r) := \operatorname{card}\left\{\, n \in \mathbb{N} \,:\, a_n^{(-)} \le r \,\right\}\,.$$

The following relations are easily verified. We have

$$\mu(r) = \sum_{k=1}^{K} \mu_k(r)\,, \tag{22}$$

and

$$a^{(-)}_{\mu(r)} \le r \quad (\forall r \in \mathbb{R})\,, \tag{23}$$

$$\mu(a^{(-)}_n) \ge n \quad (\forall n \in \mathbb{N}_0)\,. \tag{24}$$

(i) Set $C = \min\{A_1, \ldots, A_K\}$. Since for each $k \in \{1, \ldots, K\}$,

$$\left\{ n \in \mathbb{N} \,:\, a^{(k)}_n \le r \right\} \subset \{ n \in \mathbb{N} \,:\, a_k n^\alpha + C \le r \}\,,$$

we have, for $r \ge C$,

$$\mu_k(r) \le \operatorname{card}\, \{ n \in \mathbb{N} \,:\, a_k n^\alpha + C \le r \} = \left\lfloor \left(\frac{r - C}{a_k} \right)^{1/\alpha} \right\rfloor ,$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Thus, using (22), we have

$$\mu(r) = \sum_{k=1}^{K} \mu_k(r) \le (r - C)^{1/\alpha} \sum_{k=1}^{K} a_k^{-1/\alpha}\,. \tag{25}$$

If $n \in \mathbb{N}$ then $a^{(-)}_n \ge C$, so combining (25) with (24) gives

$$(a^{(-)}_n - C)^{1/\alpha} \sum_{k=1}^{K} a_k^{-1/\alpha} \ge \mu(a^{(-)}_n) \ge n\,,$$

from which (i) follows.

(ii) Set $C = \max\{A_1, \ldots, A_K\}$. Since for each $k \in \{1, \ldots, K\}$,

$$\{ n \in \mathbb{N} \,:\, a_k n^\alpha + C \le r \} \subset \left\{ n \in \mathbb{N} \,:\, a^{(k)}_n \le r \right\}\,,$$

we have, for $r \ge C$,

$$\left\lfloor \left(\frac{r - C}{a_k} \right)^{1/\alpha} \right\rfloor = \operatorname{card}\, \{ n \in \mathbb{N} \,:\, a_k n^\alpha + C \le r \} \le \mu_k(r)\,.$$

Thus, using (22), we have

$$\mu(r) = \sum_{k=1}^{K} \mu_k(r) \ge \left((r - C)^{1/\alpha} \sum_{k=1}^{K} a_k^{-1/\alpha} \right) - K\,. \tag{26}$$

Now fix $n_0 \in \mathbb{N}$. Choose $r_0 \ge C$ such that

$$n_0 = \left((r_0 - C)^{1/\alpha} \sum_{k=1}^{K} a_k^{-1/\alpha} \right) - K\,. \tag{27}$$

From (26) and (27) we see that $n_0 \le \mu(r_0)$. Using (23), together with the fact that $n \mapsto a^{(-)}_n$ is monotonically increasing, now yields

$$a^{(-)}_{n_0} \le a^{(-)}_{\mu(r_0)} \le r_0 = c(n_0 + K)^\alpha + C\,.$$

Since n_0 was arbitrary, (ii) follows. □

Corollary 5.4. *Let $\alpha > 0$ and $K \in \mathbb{N}$. Suppose that for each $k \in \{1, \dots, K\}$ we are given a real sequence $b^{(k)}$, a positive constant $a_k > 0$, and a real number B_k. Let $b^{(+)}$ denote the decreasing arrangement of the K sequences $b^{(1)}, \dots, b^{(K)}$ and define*

$$c := \left(\sum_{k=1}^{K} a_k^{-1/\alpha} \right)^{-\alpha} .$$

(i) *If*
$$b_n^{(k)} \leq B_k \exp(-a_k n^\alpha) \quad (\forall n \in \mathbb{N},\, k \in \{1, \dots, K\}) ,$$
then
$$b_n^{(+)} \leq B \exp(-c n^\alpha) \quad (\forall n \in \mathbb{N}) ,$$
where $B = \max \{B_1, \dots, B_K\}$.

(ii) *If*
$$b_n^{(k)} \geq B_k \exp(-a_k n^\alpha) \quad (\forall n \in \mathbb{N},\, k \in \{1, \dots, K\}) ,$$
then
$$b_n^{(+)} \geq B \exp(-c (n+K)^\alpha) \quad (\forall n \in \mathbb{N}) ,$$
where $B = \min \{B_1, \dots, B_K\}$.

6. Acknowledgements

I would like to thank Oliver Jenkinson for sharing his ideas, which led to some of the results in Section 2, in particular to the proof of sharpness in Proposition 2.10, and for many inspiring discussions during the preparation of this article. The research described in this article was partly supported by EPSRC Grant GR/R64650/01.

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