ANALYTIC EXPANDING CIRCLE MAPS WITH EXPLICIT SPECTRA

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ABSTRACT. We show that for any $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ there exists an analytic expanding circle map such that the eigenvalues of the associated transfer operator (acting on holomorphic functions) are precisely the nonnegative powers of λ and $\overline{\lambda}$. As a consequence we obtain a counterexample to a variant of a conjecture of Mayer on the reality of spectra of transfer operators.

1. INTRODUCTION

Transfer operators can be regarded as global representations of a system's dynamics. Their spectral properties yield insight into the statistical long-term behaviour of the underlying system, via rates of mixing or the existence of central limit theorems (see, for example, [3, 9] and refererences therein). In certain cases these mixing rates provide bounds for dynamically relevant quantities such as entropy or Lyapunov exponents, see [11, 29, 33].

If $\{\phi_k : k = 1, \ldots, K\}$ is the set of local inverse branches of a real analytic expanding map, then the associated transfer operator $\mathcal{L}$, defined by

$$(\mathcal{L}f)(z) = \sum_{k=1}^{K} \phi_k'(z)(f \circ \phi_k)(z), \tag{1}$$

preserves and acts compactly on certain spaces of holomorphic functions[1] (see, for example, [5, 7, 27, 31]). In particular, its spectrum is a sequence of eigenvalues $\{\lambda_n(\mathcal{L})\}$ converging to zero, together with zero itself.[2] Under mild assumptions it is possible to show that the spectrum does not depend on the particular choice of holomorphic function space, see [8].

Assuming that the eigenvalues of $\mathcal{L}$ are ordered with respect to decreasing modulus, the second eigenvalue $\lambda_2(\mathcal{L})$ determines the exponential rate of mixing for generic analytic observables. A faster exponential rate of mixing can occur if one chooses (non-generic) observables which do not 'feel' the rate corresponding to the first n subleading eigenvalues, that is, observables in a subspace complementary to the eigendirections of $\{\lambda_2(\mathcal{L}), \ldots, \lambda_{n+1}(\mathcal{L})\}$. Consequently, knowledge of the whole spectrum $\sigma(\mathcal{L})$ is useful, since it determines the set of all possible exponential mixing rates, also known as the correlation spectrum in this context.

As the correlation spectrum is of physical significance, there is an extensive body of literature on numerical approximation schemes for spectra of expanding systems,

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[1] Often $\mathcal{L}$ is referred to as the Perron-Frobenius operator. More general transfer operators can be obtained by replacing the ϕ_k' in (1) by other suitable weight functions.

[2] The set of non-zero eigenvalues may be finite, in which case we set $\lambda_n(\mathcal{L}) = 0$ for all large $n \in \mathbb{N}$.

which broadly fall into two categories: those with a dynamical flavour based on cycle expansions (see [2]) of the Fredholm determinant (see, for example, [10, 21]), and those with a functional analytic flavour based on finite-rank approximations of transfer operators (see, for example, [4, 17, 24]).

Surprisingly there exist only few examples of maps in the literature for which the spectrum of the corresponding transfer operator is known explicitly. The only one-dimensional examples known to the authors are piecewise linear interval maps (see, for example, [1, 18, 20]), or more generally, piecewise linear Markov interval maps (see, for example, [28] or [33] for a general account) and circle maps of the form $z \mapsto z^n$ for n a non-zero integer. It turns out that for the latter maps the spectra of the corresponding transfer operators when acting on analytic functions are all identical and coincide with the two-point set $\{0, 1\}$. See [3, Exercise 2.15] for a proof when $n = 2$; the general case can be proved along the same lines.

An interesting example with partial spectral information is given in [22], where the authors construct an analytic expanding circle map with non-trivial 'non-essential' spectrum for the transfer operator on the space of C^1 functions.

Returning to the standard set-up of this article in which the transfer operator acts compactly on spaces of holomorphic functions, the essential spectrum reduces to the origin. In this case, the eigenvalue sequence decays exponentially fast (see [16]) and it is possible to give explicit upper bounds (see [6, 7]). Lower bounds on the eigenvalue sequence have recently been obtained by Naud [30], who showed that there is a dense set of transfer operators arising from analytic expanding interval maps with infinitely many distinct non-trivial eigenvalues. To the best of our knowledge, no analogous statement for expanding maps on the circle has been established.

The purpose of this paper is twofold: firstly, to construct examples of circle maps with infinitely many explicit non-zero eigenvalues in the spectrum of $\mathcal{L}$. Denoting the circle by $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, and letting $H^\infty(A)$ be the space of bounded holomorphic functions on an annulus A containing $\mathbb{T}$ (see Section 2 for precise definitions), we can state the following.

Theorem 1.1. *For any $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ there exists an analytic expanding circle map τ such that the eigenvalues of the associated transfer operator $\mathcal{L}$ in (1), when acting on $H^\infty(A)$, are precisely all nonnegative powers of λ and $\overline{\lambda}$, that is, the spectrum of $\mathcal{L}$ is*

$$\sigma(\mathcal{L}) = \{\, \lambda^n \,:\, n \in \mathbb{N}_0 \,\} \cup \{\, \overline{\lambda}^n \,:\, n \in \mathbb{N} \,\} \cup \{0\}\,.$$

Moreover, the algebraic multiplicity of the leading eigenvalue is one, while the algebraic multiplicity of the remaining eigenvalues is two for real λ and one for λ with nonvanishing imaginary part.

Secondly, the analytic maps arising from Theorem 1.1 yield interval maps[3] which provide counterexamples to the following conjecture.

Conjecture 1.2 (Weak variant of Mayer's conjecture in dimension one)**.** Let $\Omega \subset \mathbb{C}$ be a bounded domain with $\Omega_{\mathbb{R}} = \Omega \cap \mathbb{R} \neq \emptyset$ and $\Phi_k : \Omega \to \Omega$ contracting holomorphic mappings with their unique fixed points z_k^* in $\Omega_{\mathbb{R}}$. If the $\Phi_k'(z_k^*)$ are real, then all

[3]Throughout the paper we use lower case Greek letters to denote inverse branches of circle maps and the corresponding upper case letters to denote inverse branches of interval maps.

eigenvalues of the corresponding transfer operator

$$(\mathcal{L}f)(z) = \sum_{k=1}^{K} \Phi_k'(z)(f \circ \Phi_k)(z)$$

with small enough modulus are real.

Mayer [25] originally conjectured that transfer operators satisfying the hypotheses of the above conjecture have real spectra. Counterexamples to Mayer's conjecture were given by Levin in [23] which led to the above weakening of Mayer's conjecture.

Reality of spectra has been studied in a few concrete examples. For the Gauss map, Mayer showed that the eigenvalues of the transfer operator (on an appropriately defined function space) are real and tend to zero exponentially fast, see [26] or the survey [27]. Another prominent example is the linearized Feigenbaum period doubling operator, for which numerical observations [2, 10] suggest the spectrum to be real. Furthermore, transfer operators of expanding interval maps with one 'dominating' branch have real spectra, as shown by Rugh [32] using a perturbative approach.

Before proceeding to definitions and rigorous proofs, we shall briefly explain the genesis of our result. Turning the classical question of finding eigenvalues of $\mathcal{L}$ for a given map on its head, we attempt to construct a map for which the transfer operator has a given eigenvalue and a given eigenfunction. Considering an analytic expanding map on the interval I with two full branches, the eigenvalue problem of the transfer operator formally reads

$$\mu_n u_n = \Phi_1' \cdot (u_n \circ \Phi_1) + \Phi_2' \cdot (u_n \circ \Phi_2), \tag{2}$$

where μ_n and u_n are an eigenvalue and eigenfunction of $\mathcal{L}$.

Given an analytic invariant density ρ one may consider (2) for $n = 0$ with $\mu_0 = 1$ and $u_0 = \rho$ as an equation to compute suitable inverse branches Φ_1 and Φ_2 of the map. This setup is a particularly simple case of the so called inverse Perron-Frobenius problem [15, 19], which has been applied in various guises to taylor-make chaotic maps with given stationary properties (see, for example, [13]). As we are attempting to construct a map with two branches we are at liberty to specify a nontrivial eigenvalue and corresponding eigenfunction. Thus, given an invariant density ρ, a real number λ with $|\lambda| < 1$, and a potential eigenfunction u, we seek to solve

$$\begin{aligned} P &= P \circ \Phi_1 + P \circ \Phi_2, \\ \lambda U &= U \circ \Phi_1 + U \circ \Phi_2 \end{aligned} \tag{3}$$

for the inverse branches Φ_1 and Φ_2. Here P and U denote suitable antiderivatives of ρ and u, respectively. A priori, there is no guarantee that (3) admits a real solution for Φ_1 and Φ_2 and that such a solution actually determines an analytic full branch interval map. Developing conditions under which this is the case seems to be a challenging task. Nevertheless, if we fix the interval $I = [-1, 1]$, take ρ to be the uniform density, and $u(x) = \cos(\pi x)$ for the eigenfunction with eigenvalue λ, then (3) leads to

$$\begin{aligned} x &= \Phi_1(x) + \Phi_2(x), \\ \lambda \sin(\pi x) &= \sin(\pi \Phi_1(x)) + \sin(\pi \Phi_2(x)). \end{aligned} \tag{4}$$

After a short calculation it is possible to deduce from (4) an explicit expression for Φ_1 and Φ_2, and, as we shall see in Section 3, this indeed determines an analytic full branch map. It turns out that for this particular example the complete spectrum can be obtained, an observation that provides the main content of Theorem 1.1.

Certainly, this reasoning does not generally provide examples with fully prescribed spectrum. However, the method deserves further exploration, as it furnishes maps with given partial spectral information.

The paper is structured as follows. In Section 2 we introduce analytic expanding circle maps, and define Banach spaces of holomorphic functions on which the associated transfer operators $\mathcal{L}$ are compact. Section 3 is devoted to the construction of a family of circle maps and the proof of Theorem 1.1. In Section 4 we consider the same family of maps on an interval and thus obtain counterexamples to Conjecture 1.2.

2. Transfer operators for analytic circle maps

The main purpose of this section is to define suitable function spaces on which transfer operators induced by analytic expanding circle maps are compact. We start by defining what is meant by an analytic expanding circle map.

Definition 2.1. We say that $\tau : \mathbb{T} \to \mathbb{T}$ is an *analytic expanding circle map* if the following two conditions hold:

(i) τ is analytic on $\mathbb{T}$;
(ii) $\inf_{z\in\mathbb{T}} |\tau'(z)| > 1$.

It is not difficult to see that τ is a K-fold covering of $\mathbb{T}$ for some integer $K > 1$. Moreover, the map τ has analytic extensions to certain annuli containing $\mathbb{T}$. With slight abuse of notation we shall write τ for the various extensions as well. To be precise, for $r < 1 < R$ let $A_{r,R}$ denote the open annulus $A_{r,R} = \{\, z \in \mathbb{C} : r < |z| < R \,\}$ and write

$$\mathcal{A} = \{\, A_{r,R} \,:\, \tau \text{ is analytic on } A_{r,R} \,\}\,.$$

The expansivity of τ yields the following result.

Lemma 2.2. *If τ is an analytic expanding circle map, then there is $A_0 \in \mathcal{A}$ such that*

(a) both τ and $1/\tau$ are analytic on the closure $\mathrm{cl}(A_0)$ of A_0;
(b) $\tau(\partial A_0) \cap \mathrm{cl}(A_0) = \emptyset$, where ∂A_0 denotes the boundary of A_0.

Proof. Since τ is an analytic expanding circle map it is possible to choose $A_1 \in \mathcal{A}$ such that both τ and $1/\tau$ are analytic on $\mathrm{cl}(A_1)$ with

$$\alpha := \inf_{z\in A_1} |\tau'(z)| > 1\,.$$

It is not difficult to see that $(\rho,\theta) \mapsto \log\left|\tau(\rho e^{i\theta})\right|$ is differentiable for all (ρ,θ) with $\rho\exp(i\theta) \in A_1$ and

$$\frac{\partial}{\partial\rho}\log\left|\tau(\rho e^{i\theta})\right| = \Re\left(e^{i\theta}\frac{\tau'(\rho e^{i\theta})}{\tau(\rho e^{i\theta})}\right)\,, \tag{5}$$

$$\frac{\partial}{\partial\theta}\log\left|\tau(\rho e^{i\theta})\right| = -\Im\left(\rho e^{i\theta}\frac{\tau'(\rho e^{i\theta})}{\tau(\rho e^{i\theta})}\right)\,, \tag{6}$$

where $\Re(z)$ and $\Im(z)$ denote the real and imaginary part of $z \in \mathbb{C}$. Since τ leaves $\mathbb{T}$ invariant, equation (6) implies either

$$e^{i\theta}\frac{\tau'(e^{i\theta})}{\tau(e^{i\theta})} \geq \alpha \quad (\forall \theta \in \mathbb{R}), \tag{7}$$

or

$$e^{i\theta}\frac{\tau'(e^{i\theta})}{\tau(e^{i\theta})} \leq -\alpha \quad (\forall \theta \in \mathbb{R}). \tag{8}$$

Suppose now that (7) holds (the other case can be dealt with similarly). Fixing β with $1 < \beta < \alpha$ we can choose $A_{r,R} \in \mathcal{A}$ with $A_{r,R} \subset A_1$ and $e^{\beta(r-1)} < r$, $e^{\beta(R-1)} > R$ such that

$$\Re\left(e^{i\theta}\frac{\tau'(\rho e^{i\theta})}{\tau(\rho e^{i\theta})}\right) \geq \beta \quad (\forall \rho \in [r,R], \forall \theta \in \mathbb{R}).$$

Equation (5) now implies

$$\log\left|\tau(e^{i\theta})\right| - \log\left|\tau(re^{i\theta})\right| = \Re\int_r^1 e^{i\theta}\frac{\tau'(\rho e^{i\theta})}{\tau(\rho e^{i\theta})}\,d\rho \geq \beta(1-r)$$

and

$$\log\left|\tau(Re^{i\theta})\right| - \log\left|\tau(e^{i\theta})\right| = \Re\int_1^R e^{i\theta}\frac{\tau'(\rho e^{i\theta})}{\tau(\rho e^{i\theta})}\,d\rho \geq \beta(R-1).$$

Thus

$$\left|\tau(re^{i\theta})\right| \leq e^{\beta(r-1)} < r \qquad \text{and} \qquad \left|\tau(Re^{i\theta})\right| \geq e^{\beta(R-1)} > R,$$

so $A_0 := A_{r,R}$ has all the desired properties. □

Given an expanding circle map τ, we associate with it a transfer operator $\mathcal{L}$ by setting

$$(\mathcal{L}f)(z) = \sum_{k=1}^{K}\phi_k'(z)(f\circ\phi_k)(z) \quad (z \in \mathbb{T}), \tag{9}$$

where ϕ_k to denotes the k-th local inverse of τ. It turns out that $\mathcal{L}$ is well-defined and bounded as an operator on $L^1(\mathbb{T})$. Although this is a standard result, we shall provide a short proof, since part of the argument will play a crucial role later on. In the following we shall use $\mathcal{C}_\rho$ to denote a simple closed positively oriented path along the circle centred at the origin with radius ρ.

Lemma 2.3. *For any $f \in L^1(\mathbb{T})$ and any $g \in L^\infty(\mathbb{T})$ we have*

$$\frac{1}{2\pi i}\int_{\mathcal{C}_1}(\mathcal{L}f)(z)\cdot g(z)\,dz = \frac{1}{2\pi i}\int_{\mathcal{C}_1} f(z)\cdot(g\circ\tau)(z)\,dz. \tag{10}$$

In particular, the transfer operator $\mathcal{L}$ is bounded as an operator from $L^1(\mathbb{T})$ to $L^1(\mathbb{T})$.

Proof. Using change of variables we see that

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mathcal{C}_1}(\mathcal{L}f)(z)\cdot g(z)\,dz &= \sum_{k=1}^{K}\frac{1}{2\pi i}\int_{\mathcal{C}_1}\phi_k'(z)\cdot(f\circ\phi_k)(z)\cdot g(z)\,dz\\ &= \sum_{k=1}^{K}\frac{1}{2\pi i}\int_{\phi_k(\mathcal{C}_1)} f(z)\cdot(g\circ\tau)(z)\,dz = \frac{1}{2\pi i}\int_{\mathcal{C}_1} f(z)\cdot(g\circ\tau)(z)\,dz,\end{aligned}$$

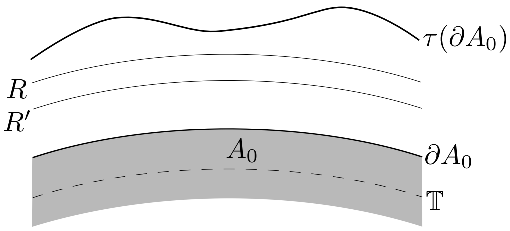

FIGURE 1. Proof of Lemma 2.5: choice of annuli A_0, A' and A.

where we have used the fact that $\cup_{k=1}^{K}\phi_k(\mathcal{C}_1) = \mathcal{C}_1$ up to a set of measure zero. Finally, the assertion that $\mathcal{L}$ maps $L^1(\mathbb{T})$ continuously into itself follows from equation (10), since for any $f \in L^1(\mathbb{T})$ and any $g \in L^\infty(\mathbb{T})$

$$\left|\frac{1}{2\pi i}\int_{\mathcal{C}_1}(\mathcal{L}f)(z)\cdot g(z)\,dz\right| \leq \|f\|_{L^1(\mathbb{T})}\,\|g\circ\tau\|_{L^\infty(\mathbb{T})} = \|f\|_{L^1(\mathbb{T})}\,\|g\|_{L^\infty(\mathbb{T})}\,. \qquad \square$$

It turns out that $\mathcal{L}$ leaves certain subspaces of $L^1(\mathbb{T})$ invariant. Of particular interest are spaces consisting of holomorphic functions.

Definition 2.4. For U, an open subset of $\mathbb{C}$, we write

$$H^\infty(U) = \{\, f : U \to \mathbb{C} \,:\, f \text{ holomorphic and } \sup_{z\in U}|f(z)| < \infty \,\}$$

to denote the Banach space of bounded holomorphic functions on U equipped with the norm $\|f\|_{H^\infty(U)} = \sup_{z\in U}|f(z)|$.

The proof of the invariance of $H^\infty(U)$ under $\mathcal{L}$ for suitable U will rely on Fourier theory. Here and in the following, we shall use

$$c_n(f) = \frac{1}{2\pi i}\int_{\mathcal{C}_1}\frac{f(z)}{z^{n+1}}\,dz \tag{11}$$

to denote the n-th Fourier coefficient of $f \in L^1(\mathbb{T})$.

Before stating the next result we require some more notation. Given two subsets U and V of $\mathbb{C}$ we write

$$U \Subset V$$

if $\mathrm{cl}(U)$ is a compact subset of V.

We now have the following result.

Lemma 2.5. *Suppose that annuli A and A' in $\mathcal{A}$ have been chosen[4] such that*

$$A_0 \Subset A' \subset A \text{ and } \tau(\partial A_0) \cap \mathrm{cl}(A) = \emptyset\,. \tag{12}$$

Then the transfer operator $\mathcal{L}$ maps $H^\infty(A')$ continuously to $H^\infty(A)$.

Proof. Given $f \in H^\infty(A')$, we shall show that $\mathcal{L}f \in H^\infty(A)$ by estimating the asymptotic behaviour of the Fourier coefficients of $\mathcal{L}f$. Write R_0 and R to denote

[4]This is always possible by Lemma 2.2.

the radii of the circles forming the 'exterior' boundary of A_0 and A, respectively (see Figure 1). Next choose R'' with

$$\inf_{z\in\mathcal{C}_{R_0}} |\tau(z)| > R'' > R\,.$$

Similarly, write r_0 and r to denote the radii of the circles forming the 'interior' boundary of A_0 and A, respectively and choose r'' with

$$\sup_{z\in\mathcal{C}_{r_0}} |\tau(z)| < r'' < r\,.$$

Fix $f \in H^\infty(A')$ with $\|f\|_{H^\infty(A')} \leq 1$ and let $n \geq 0$. Using Lemma 2.3 we see that

$$\begin{aligned}|c_n(\mathcal{L}f)| = \left|\frac{1}{2\pi i}\int_{\mathcal{C}_1}\frac{f(z)}{\tau(z)^{n+1}}\,dz\right| &= \left|\frac{1}{2\pi i}\int_{\mathcal{C}_{R_0}}\frac{f(z)}{\tau(z)^{n+1}}\,dz\right|\\ &\leq \frac{1}{2\pi}\int_{\mathcal{C}_{R_0}}\frac{1}{|\tau(z)|^{n+1}}\,|dz| \leq \frac{R_0}{(R'')^{n+1}}\,.\end{aligned}$$

Similarly, for $n \geq 1$ we have

$$\begin{aligned}|c_{-n}(\mathcal{L}f)| = \left|\frac{1}{2\pi i}\int_{\mathcal{C}_1}\frac{f(z)}{\tau(z)^{-n+1}}\,dz\right| &= \left|\frac{1}{2\pi i}\int_{\mathcal{C}_{r_0}} f(z)\tau(z)^{n-1}\,dz\right|\\ &\leq \frac{1}{2\pi}\int_{\mathcal{C}_{r_0}}|\tau(z)|^{n-1}\,|dz| \leq r_0(r'')^{n-1}\,.\end{aligned}$$

Hence, $\sum_{n=0}^\infty c_n(\mathcal{L}f)z^n$ converges absolutely for all $|z| \leq R$ and $\sum_{n=1}^\infty c_{-n}(\mathcal{L}f)z^{-n}$ converges absolutely for all $|z| \geq r$. Moreover, for $z \in A$ we have

$$\begin{aligned}\left|\sum_{n=-\infty}^{\infty} c_n(\mathcal{L}f)z^n\right| &\leq \sum_{n=0}^\infty |c_n(\mathcal{L}f)|\,R^n + \sum_{n=1}^\infty |c_{-n}(\mathcal{L}f)|\,r^{-n}\\ &\leq \sum_{n=0}^\infty \frac{R_0R^n}{(R'')^{n+1}} + \sum_{n=1}^\infty \frac{r_0(r'')^{n-1}}{r^n} = \frac{R_0}{R''-R} + \frac{r_0}{r-r''}\,.\end{aligned}$$

Thus, by the uniqueness of the Fourier transform on $L^1(\mathbb{T})$, we conclude that $\mathcal{L}f \in H^\infty(A)$ and

$$\|\mathcal{L}f\|_{H^\infty(A)} \leq \left(\frac{R_0}{R''-R} + \frac{r_0}{r-r''}\right)\|f\|_{H^\infty(A')} \quad (\forall f \in H^\infty(A'))\,. \qquad \square$$

Choosing $A = A'$ in the previous lemma shows that $\mathcal{L}$ induces a well defined continuous operator from $H^\infty(A)$ to itself. It turns out that $\mathcal{L} : H^\infty(A) \to H^\infty(A)$ is compact. In order to prove this result, we shall employ a factorisation argument.

Given $A, A' \in \mathcal{A}$ with $A' \Subset A$ define the canonical embedding $J : H^\infty(A) \to H^\infty(A')$

$$(Jf) = f|_{A'}\,. \tag{13}$$

The embedding J is compact by Montel's Theorem (see, for example, [12, Chapter 7, Theorem 2.9]) and, as we shall shortly see, is well-approximated by the following operators: for N a positive integer, define the finite rank operator $J_N : H^\infty(A) \to H^\infty(A')$ by

$$(J_N f)(z) = \sum_{n=-N-1}^{N-1} c_n(f) z^n \quad \text{for } z \in A' . \tag{14}$$

The approximability of J alluded to earlier is the content of the following result.

Lemma 2.6. *Let J and J_N be defined as above. Then*

$$\lim_{N\to\infty} \|J - J_N\|_{H^\infty(A)\to H^\infty(A')} = 0 .$$

In particular, the embedding J is compact.

Proof. Choose $A'' \in \mathcal{A}$ with

$$A' \subset\subset A'' \subset\subset A .$$

Let R' and r' denote the radii of the circle forming the 'exterior' and 'interior' boundary of A', respectively, and let $\mathcal{C}_{R''}$ and $\mathcal{C}_{r''}$ denote the oriented 'exterior' and 'interior' boundary of A'', respectively, so that

$$r'' < r' < R' < R'' .$$

Fix $f \in H^\infty(A)$ with $\|f\|_{H^\infty(A)} \leq 1$. Then

$$\begin{aligned}
\|Jf - J_N f\|_{H^\infty(A')} &= \sup_{z\in A'} \left| \sum_{n\geq N} \frac{z^n}{2\pi i} \int_{\mathcal{C}_1} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta + \sum_{n\geq N+2} \frac{z^{-n}}{2\pi i} \int_{\mathcal{C}_1} \frac{f(\zeta)}{\zeta^{-n+1}}\, d\zeta \right| \\
&\leq \sum_{n\geq N} \frac{(R')^n}{2\pi} \int_{\mathcal{C}_{R''}} \frac{|f(\zeta)|}{|\zeta|^{n+1}} |d\zeta| + \sum_{n\geq N+2} \frac{(r')^{-n}}{2\pi} \int_{\mathcal{C}_{r''}} \frac{|f(\zeta)|}{|\zeta|^{-n+1}} |d\zeta| \\
&\leq \left(\frac{R'}{R''}\right)^N \frac{1}{1-\frac{R'}{R''}} + \left(\frac{r''}{r'}\right)^{N+2} \frac{1}{1-\frac{r''}{r'}}
\end{aligned}$$

from which the assertions follow. □

Now, the transfer operator $\mathcal{L} : H^\infty(A) \to H^\infty(A)$, factorises as

$$\mathcal{L} = \tilde{\mathcal{L}} J \tag{15}$$

where $J : H^\infty(A) \to H^\infty(A')$ is the canonical embedding, which is compact by Lemma 2.6, and $\tilde{\mathcal{L}}$ is the transfer operator viewed as an operator from $H^\infty(A')$ to $H^\infty(A)$, guaranteed to be continuous by Lemma 2.5. Thus, the factorisation (15) implies the following result.

Proposition 2.7. *Let $A \in \mathcal{A}$ with $A_0 \subset\subset A$. Then $\mathcal{L} : H^\infty(A) \to H^\infty(A)$ is compact.*

3. A family of circle maps

In this section we introduce a family of analytic expanding circle maps for which we are able to explicitly determine the spectrum of $\mathcal{L}$. As already mentioned in the introduction, the family arises from the construction of an expanding map for which the transfer operator has a specified eigenfunction for a given eigenvalue λ. A short calculation reveals that for $\lambda \in (-1, 1)$ a solution to (4) lifts to $\Phi\colon \mathbb{R} \to \mathbb{R}$, where

$$\Phi(x) = \frac{x}{2} - \frac{1}{\pi} \arccos\left(\lambda \cos\left(\frac{\pi x}{2}\right)\right) . \tag{16}$$

This is an increasing diffeomorphism with inverse $F\colon \mathbb{R} \to \mathbb{R}$ given by

$$F(x) = 2x + 1 + \frac{2}{\pi} \arctan\left(\frac{\lambda \sin(\pi x)}{1 - \lambda \cos(\pi x)}\right). \tag{17}$$

Note that F is a lift of a circle map $\tau : \mathbb{T} \to \mathbb{T}$, which satisfies $F(x+2) = F(x) + 4$ and $p \circ F = \tau \circ p$, where $p : \mathbb{R} \to \mathbb{T}$ is the projection map defined by $p(x) = e^{i\pi x}$. The map τ is a twofold covering of $\mathbb{T}$ (see Figure 2). Note that $F' > 1$ for $\lambda \in (-1, 1)$. Thus τ is an analytic expanding circle map.

It turns out that τ can be written in closed form. Using the relation $e^{i\pi F(x)} = \tau(e^{i\pi x})$, it follows that, for $\lambda \in \mathbb{R}$,

$$\pi F(x) = \pi x + \arg(\frac{\lambda - e^{i\pi x}}{1 - \lambda e^{i\pi x}}) = 2\pi x + \pi + 2 \arg(1 - \lambda e^{-i\pi x}),$$

which gives

$$\tau(z) = \frac{z(\lambda - z)}{1 - \bar{\lambda} z} \quad \text{for } z \in \mathbb{T}. \tag{18}$$

It is not difficult to see that the above expression for τ yields an analytic expanding circle map not just for real λ, but for any $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ (see Figure 2). It is possible to write down lifts of (18) for complex λ similar to (17). In fact, a short calculation shows that if $\lambda = |\lambda| e^{i\alpha}$ then the argument of arctan in (17) needs to be replaced by $|\lambda| \sin(\pi x - \alpha)/(1 - |\lambda| \cos(\pi x - \alpha))$. Given τ as in (18), we now

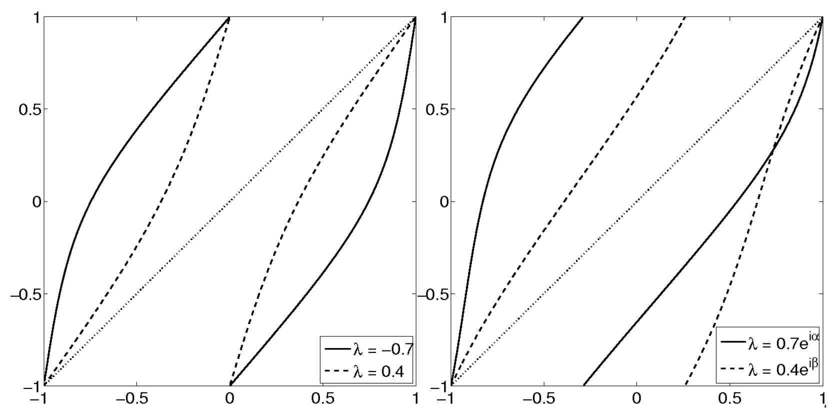

FIGURE 2. τ projected on the interval $[-1, 1]$ for (left) $\lambda = -0.7$ and $\lambda = 0.4$ and (right) $\lambda = -0.3 - i\sqrt{0.4} = 0.7e^{i\alpha}$ with $\alpha \approx -2.0137$ and $\lambda = 0.1 + i\sqrt{0.15} = 0.4e^{i\beta}$ with $\beta \approx 1.318$. Note that the projection is chosen such that the interval endpoint -1 is fixed by τ.

choose an annulus $A \in \mathcal{A}$ with $A_0 \subset\subset A$. By Proposition 2.7 the associated transfer operator $\mathcal{L} : H^\infty(A) \to H^\infty(A)$ is well-defined and compact.

As we shall see, the spectrum of $\mathcal{L}$ can be computed by analysing the spectrum of a suitable matrix representation, which is obtained as follows. For $N \in \mathbb{N}$ consider the projection P_N given by the same functional expression as J_N in (14), now viewed as an operator from $H^\infty(A)$ to $H^\infty(A)$. Clearly, $P_N \mathcal{L} P_N$ is an operator of rank $2N+1$. Writing $e_n(z)=z^n$, the set $\{ e_n : -N-1 \leq n \leq N-1 \}$ is a basis for

$$H_N = P_N(H^\infty(A))$$

and the restriction of $P_N \mathcal{L} P_N$ to H_N is represented by the $(2N+1)\times(2N+1)$ matrix $L^{(N)}$ defined by

$$(L^{(N)})_{n,l} = c_{n-1}(\mathcal{L} e_{l-1}) = \frac{1}{2\pi i}\int_{\mathcal{C}_1} \frac{z^l}{\tau(z)^n}\frac{dz}{z} = \frac{1}{2\pi i}\int_{\mathcal{C}_1} z^{l-n}\Big(\frac{1-\overline{\lambda}z}{\lambda - z}\Big)^n \frac{dz}{z}. \quad (19)$$

In particular, the non-zero spectrum of $P_N \mathcal{L} P_N$ is given by the non-zero spectrum of $L^{(N)}$.

Observe that (19) defines an infinite matrix L containing $L^{(N)}$ as a finite submatrix. The following lemma summarizes the properties of L.

Lemma 3.1. *For $l, n \in \mathbb{Z}$ the following hold:*

(a) $L_{0,0} = 1$;
(b) $L_{0,l} = 0$ *if* $l \neq 0$;
(c) $L_{-n,-l} = \overline{L_{n,l}}$;
(d) $L_{-n,-n} = \lambda^n$ *for* $n \geq 0$;
(e) $L_{n,l} = L_{-n,-l} = 0$ *for* $n > l$.

Proof. Assertions (a) and (b) immediately follow from (19),
while (c) is a consequence of

$$\begin{aligned} L_{-n,-l} &= \frac{1}{2\pi i}\int_{\mathcal{C}_1} z^{n-l}\Big(\frac{\lambda - z}{1-\overline{\lambda}z}\Big)^n \frac{dz}{z} = \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta(n-l)}\Big(\frac{\lambda - e^{i\theta}}{1-\overline{\lambda}e^{i\theta}}\Big)^n d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta(l-n)}\Big(\frac{1-\lambda e^{-i\theta}}{\overline{\lambda} - e^{-i\theta}}\Big)^n d\theta = \overline{L_{n,l}}\,. \end{aligned}$$

For (d) and (e), observe that $z \mapsto \left(\frac{\lambda - z}{1-\overline{\lambda}z}\right)$ is holomorphic for all z in the closed unit disk. Thus, by the Residue Theorem,

$$L_{-n,-n} = \frac{1}{2\pi i}\int_{\mathcal{C}_1} \frac{1}{z}\Big(\frac{\lambda - z}{1-\overline{\lambda}z}\Big)^n dz = \lambda^n.$$

Finally $n > l$ implies $L_{-n,-l} = 0$, as the intergrand is a holomorphic function. ☐

The lemma above implies that $L^{(N)}$ has the following upper-lower triangular matrix structure[5]

$$L^{(N)} = \begin{pmatrix} \lambda^N & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & 0 & 0 & \vdots & \ddots & \vdots \\ * & * & \lambda & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & \overline{\lambda} & * & * \\ \vdots & \ddots & \vdots & 0 & 0 & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & 0 & \overline{\lambda}^N \end{pmatrix}.$$

Clearly, the spectrum of $L^{(N)}$ is given by the diagonal elements $(L^{(N)})_{n,n}$, that is,

$$\sigma(L^{(N)}) = \{\lambda^n : n = 0, \dots, N\} \cup \{\overline{\lambda}^n : n = 1, \dots, N\}.$$

Moreover, the triangular structure of $L^{(N)}$ implies $\mathcal{L}(H_N) \subseteq H_N$. Before embarking on the proof of our main result, we require one more lemma, which relates the eigenvalues of $\mathcal{L}$ with the eigenvalues of $L^{(N)}$.

Lemma 3.2. *Let $A \in \mathcal{A}$ with $A_0 \subset\subset A$ and suppose that $\mathcal{L}(H_N) \subseteq H_N$ for every $N \in \mathbb{N}$. Then the non-zero eigenvalues (with multiplicities) of $\mathcal{L} : H^\infty(A) \to H^\infty(A)$ are precisely the non-zero eigenvalues of $L^{(N)}$ as N tends to infinity. In particular, the spectrum of $\mathcal{L}$ is given by*

$$\sigma(\mathcal{L}) = \mathrm{cl}(\bigcup_{N\in\mathbb{N}} \sigma(\mathcal{L}|_{H_N})) = \mathrm{cl}(\bigcup_{N\in\mathbb{N}} \sigma(L^{(N)}))\,.$$

Proof. Clearly $\sigma(L^{(N)}) \subseteq \sigma(\mathcal{L})$. Since $\mathcal{L}(H_N) \subseteq H_N$ we have $\mathcal{L}P_N = P_N\mathcal{L}P_N$ for every $N \in \mathbb{N}$. Using the factorisation (15) we see that

$$\begin{aligned} \|\mathcal{L} - P_N\mathcal{L}P_N\|_{H^\infty(A)\to H^\infty(A)} &= \|\mathcal{L} - \mathcal{L}P_N\|_{H^\infty(A)\to H^\infty(A)} \\ &= \|\tilde{\mathcal{L}}J - \tilde{\mathcal{L}}JP_N\|_{H^\infty(A)\to H^\infty(A)} \\ &= \|\tilde{\mathcal{L}}(J - J_N)\|_{H^\infty(A)\to H^\infty(A)} \\ &\le \|\tilde{\mathcal{L}}\|_{H^\infty(A')\to H^\infty(A)}\|(J - J_N)\|_{H^\infty(A)\to H^\infty(A')}\,, \end{aligned}$$

which, using Lemmas 2.5 and 2.6, implies

$$\lim_{N\to\infty} \|\mathcal{L} - P_N\mathcal{L}P_N\|_{H^\infty(A)\to H^\infty(A)} = 0\,.$$

This, together with [14, XI.9.5] guarantees that every non-zero eigenvalue of $\mathcal{L}$ is an eigenvalue of $L^{(N)}$ for some $N \in \mathbb{N}$. □

We are now able to prove our main result.

---

[5]Note that the matrix elements can be computed explicitly. For $n > 0$ and $l \in \mathbb{Z}$ we have

$$(L^{(N)})_{n,l} = \begin{cases} (-1)^{n-l}(\overline{\lambda})^{2n-l} \sum_{m=0}^{l-n} \binom{l-m-1}{n-1}\binom{n}{m}(-|\lambda|^2)^{l-n-m} & \text{if } l \le 2n \\ (-1)^n \lambda^{l-2n} \sum_{m=0}^{n} \binom{l-m-1}{n-1}\binom{n}{m}(-|\lambda|^2)^{n-m} & \text{if } l \ge 2n. \end{cases}$$

Proof of Theorem 1.1. For every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, the map τ in (18) is an analytic expanding circle map. By Lemma 3.2 the spectrum of the associated $\mathcal{L} : H^\infty(A) \to H^\infty(A)$ consists of eigenvalues, together with zero, given by

$$\sigma(\mathcal{L}) = \mathrm{cl}(\bigcup \sigma(\mathcal{L}|_{H_N})) = \mathrm{cl}(\bigcup \sigma(L^{(N)})) = \{\lambda^n : n \in \mathbb{N}_0\} \cup \{\overline{\lambda}^n : n \in \mathbb{N}\} \cup \{0\}\,.$$

The assertions concerning the multiplicities of the eigenvalues of $\mathcal{L}$ follow from the corresponding properties of $L^{(N)}$. □

4. Circle maps considered on an interval

In the previous section we have considered the transfer operator $\mathcal{L}_{\mathbb{T}} : H^\infty(A) \to H^\infty(A)$ associated to an analytic expanding circle map $\tau : \mathbb{T} \to \mathbb{T}$, which maps the space of bounded holomorphic functions on an appropriately chosen annulus $A \in \mathcal{A}$ compactly into itself. The circle map τ gives rise to a map T on an interval $I = [x_0, x_1]$, chosen such that a fixed point z_0 of τ corresponds to the interval endpoint x_0. Choosing a suitable complex neighbourhood D of I, we shall now study the spectral properties of $\mathcal{L}_I : H^\infty(D) \to H^\infty(D)$, the transfer operator corresponding to T.

More precisely, let $T : I \to I$ denote the interval map arising from the circle map $\tau : \mathbb{T} \to \mathbb{T}$ via $p \circ T = \tau \circ p$ with a projection $p : I \to \mathbb{T}$ satisfying $p(x_0) = z_0$.[6] Let $\{\Phi_k : k = 1, \ldots, K\}$ be the set of inverse branches of T. With slight abuse of notation we keep writing T and Φ_k for the respective analytic extensions to neighborhoods containing I. Since τ is an analytic K-covering, we have the matching conditions (with suitable labelling of the inverse branches)

$$\begin{aligned} &\Phi_1(x_0) = x_0\,, && \Phi_K(x_1) = x_1\,, && \Phi_1^{(n)}(x_0) = \Phi_K^{(n)}(x_1)\,, \\ &\Phi_{k+1}(x_0) = \Phi_k(x_1)\,, && \Phi_{k+1}^{(n)}(x_0) = \Phi_k^{(n)}(x_1) && \text{for } k = 1, \ldots, K-1\,, \end{aligned} \tag{20}$$

where for each $n \in \mathbb{N}$, we use $\Phi_k^{(n)}$ to denote the n-th derivative of Φ_k.

Since T is expanding, all inverse branches Φ_k are contractions. We can thus choose a topological disk D containing I such that $p(D) = A$ and $\Phi_k(D) \Subset D$ for all k. Then $\mathcal{L}_I : H^\infty(D) \to H^\infty(D)$, given by

$$\mathcal{L}_I f = \sum_{k=1}^{K} \Phi_k' \cdot (f \circ \Phi_k)\,,$$

yields a bounded operator. Moreover, $\mathcal{L}_I$ is compact (see, for example, [7, 27]), its spectrum consisting of countably many eigenvalues accumulating at zero only.

Remark 4.1. It is perhaps not surprising that the operators $\mathcal{L}_{\mathbb{T}}$ and $\mathcal{L}_I$ are closely related. In order to see this, we define the operator $Q_p : H^\infty(A) \to H^\infty(D)$ by

$$(Q_p f)(x) = p'(x) f(p(x))\,.$$

Clearly $p(D) = A$ implies Q_p injective. However, the operator Q_p is not surjective, as the image $\mathrm{im}(Q_p) = \{f \in H^\infty(D) : f^{(n)}(x_0) = f^{(n)}(x_1)\ \forall n \in \mathbb{N}_0\}$ is not all of $H^\infty(D)$. It is easy to verify that $\mathcal{L}_I$ and $\mathcal{L}_{\mathbb{T}}$ are related by

$$\mathcal{L}_I Q_p = Q_p \mathcal{L}_{\mathbb{T}}\,,$$

[6] A suitable choice is $p(x) = e^{2i\pi \frac{(x-x_0)}{(x_1-x_0)} + i \arg z_0}$.

and that $\sigma(\mathcal{L}_{\mathbb{T}}) \subseteq \sigma(\mathcal{L}_I)$, which follows from the injectivity of Q_p. On the other hand, an eigenvalue of $\mathcal{L}_I$ with an eigenfunction f is also an eigenvalue of $\mathcal{L}_{\mathbb{T}}$ if $f \in \operatorname{im}(Q_p)$.

The following lemma connects the spectrum of $\mathcal{L}_I$ with the spectrum of $\mathcal{L}_{\mathbb{T}}$. This result is mentioned in the introduction of [22] together with a proof based on the theory of Fredholm determinants. Here we shall give a short alternative proof.

Lemma 4.2. *Suppose that τ is an analytic expanding circle map and $T : I \to I$ the corresponding interval map fixing the interval endpoint x_0. Let $\mathcal{L}_{\mathbb{T}}$ and $\mathcal{L}_I$ be the corresponding transfer operators as defined above. Then the spectrum of $\mathcal{L}_I$ is given by*

$$\sigma(\mathcal{L}_I) = \sigma(\mathcal{L}_{\mathbb{T}}) \cup \{ (T'(x_0))^{-n} \, : \, n \in \mathbb{N} \} .$$

Proof. Let $H^\infty(D)^*$ denote the strong dual of $H^\infty(D)$ and let $\mathcal{L}_I^* : H^\infty(D)^* \to H^\infty(D)^*$ denote the dual operator of $\mathcal{L}_I$, that is,

$$(\mathcal{L}_I^* l)(f) = l(\mathcal{L}_I f) \quad (\forall l \in H^\infty(D)^*, \forall f \in H^\infty(D)) .$$

For $n \in \mathbb{N}_0$, let $l_n \in H^\infty(D)^*$ be defined by

$$l_n(f) = f^{(n)}(x_1) - f^{(n)}(x_0) \quad \text{for } f \in H^\infty(D) .$$

It is not difficult to see that l_0 is an eigenvector of $\mathcal{L}^*$ with eigenvalue $\Phi_1^{'}(x_0)$ since

$$\begin{aligned}
(\mathcal{L}_I^* l_0)(f) =& (\mathcal{L}_I f)(x_1) - (\mathcal{L}_I f)(x_0) \\
=& \Phi_K'(x_1)(f \circ \Phi_K)(x_1) - \Phi_1'(x_0)(f \circ \Phi_1)(x_0) \\
&+ \sum_{k=1}^{K-1} \Phi_k'(x_1)(f \circ \Phi_k)(x_1) - \Phi_{k+1}'(x_0)(f \circ \Phi_{k+1})(x_0) \\
=& \Phi_1'(x_0)(f(x_1) - f(x_0)) \\
=& \Phi_1'(x_0) l_0(f) ,
\end{aligned}$$

where the penultimate equality follows from (20). We can proceed similarly for an arbitrary $n \in \mathbb{N}$. Observe that the n-th derivative of $\mathcal{L}_I f$ is given by

$$(\mathcal{L}_I f)^{(n)} = \sum_{k=1}^{K} \sum_{m=0}^{n-1} w_{k,m} \cdot (f^{(m)} \circ \Phi_k) + \sum_{k=1}^{K} (\Phi_k')^{n+1} \cdot (f^{(n)} \circ \Phi_k),$$

where each $w_{k,m}$ is a weight function composed of derivatives of Φ_k of order up to $m+1$, and in analogy with (20) satisfying $w_{k,m}(x_1) = w_{k+1,m}(x_0)$ for $k = 1, \dots, K-1$. A calculation similar to the above yields

$$\begin{aligned}
(\mathcal{L}_I^* l_n)(f) =& (\mathcal{L}_I f)^{(n)}(x_1) - (\mathcal{L}_I f)^{(n)}(x_0) \\
=& \sum_{m=0}^{n-1} w_{1,m}(x_0) l_m(f) + (\Phi_1'(x_0))^{n+1} l_n(f).
\end{aligned}$$

It follows that $\mathcal{L}_I^* V_n \subseteq V_n$, where $V_n = \operatorname{span}\{l_0, \dots, l_n\}$ for each n. Thus $(\Phi_1'(x_0))^n$ is an eigenvalue of $\mathcal{L}_I^*$, and hence of $\mathcal{L}_I$. As $T'(x_0) = 1/\Phi_1'(x_0)$ and every eigenvalue of $\mathcal{L}_{\mathbb{T}}$ is an eigenvalue of $\mathcal{L}_I$, we have shown

$$\sigma(\mathcal{L}_{\mathbb{T}}) \cup \{T'(x_0)^{-n} : n \in \mathbb{N}\} \subseteq \sigma(\mathcal{L}_I).$$

For the converse inclusion recall Remark 4.1 and assume that $f \in H^\infty(D)$ is an eigenfunction of $\mathcal{L}_I$ with eigenvalue μ and $f \notin \operatorname{im}(Q_p)$. It follows that there is

$N \in \mathbb{N}_0$ such that $f^{(N)}(x_0) \neq f^{(N)}(x_1)$ and $f^{(n)}(x_0) = f^{(n)}(x_1)$ for $0 \leq n < N$, from which $l_n(f) = 0$ for $0 \leq n < N$. Since $\mathcal{L}_I f = \mu f$, this implies

$$l_N(\mu f) = l_N(\mathcal{L}_I f) = (\Phi_1'(x_0))^{N+1} l_N(f) .$$

As l_N is linear and non-zero, it follows that $\mu = (\Phi_1'(x_0))^{N+1} = (T'(x_0))^{-N-1}$. □

We can now apply this result to the interval maps arising from the family of circle maps defined in Section 3. Let $I = [-1, 1]$ and $\lambda \in \mathbb{R}$ with $|\lambda| < 1$, then the interval map T arising from τ in (18) fixes the interval endpoint $x_0 = -1$ with $1/T'(-1) = (\lambda + 1)/2$. By Theorem 1.1 and Lemma 4.2, the eigenvalues of $\mathcal{L}_I$ can be divided into two classes, those given by the eigenvalues of $\mathcal{L}_{\mathbb{T}}$ (each of multiplicity two, except the eigenvalue 1 of multiplicity one) and those by the powers of the inverse multiplier of the fixed point x_0, that is,

$$\sigma(\mathcal{L}_I) = \Big(\{\lambda^n : n \in \mathbb{N}_0\} \cup \{0\}\Big) \cup \{\Big(\frac{\lambda+1}{2}\Big)^n : n \in \mathbb{N}\} , \tag{21}$$

see also Figure 3.

Considering $\lambda \in \mathbb{C}$, say $\lambda = |\lambda| e^{i\alpha}$ with $|\lambda| < 1$, we now obtain counterexamples to Conjecture 1.2. The fixed point of τ is $z_0 = (\lambda - 1)/(1 - \overline{\lambda}) \in \mathbb{T}$ with

$$T'(-1) = \tau'(z_0) = \frac{\lambda + \overline{\lambda} - 2}{\lambda\overline{\lambda} - 1} = \frac{2(|\lambda|\cos(\alpha) - 1)}{|\lambda|^2 - 1} .$$

As above, the spectrum of $\mathcal{L}_I$ splits into two parts:

$$\sigma(\mathcal{L}_I) = \Big(\{\lambda^n : n \in \mathbb{N}_0\} \cup \{\overline{\lambda}^n : n \in \mathbb{N}\} \cup \{0\}\Big) \cup \{\Big(\frac{|\lambda|^2 - 1}{2(|\lambda|\cos(\alpha) - 1)}\Big)^n : n \in \mathbb{N}\}.$$

Note that the transfer operator $\mathcal{L}_I$ associated to T satisfies the conditions of Conjecture 1.2, but, for $\lambda \notin \mathbb{R}$, has countably infinitely many non-real eigenvalues of arbitrarily small modulus.

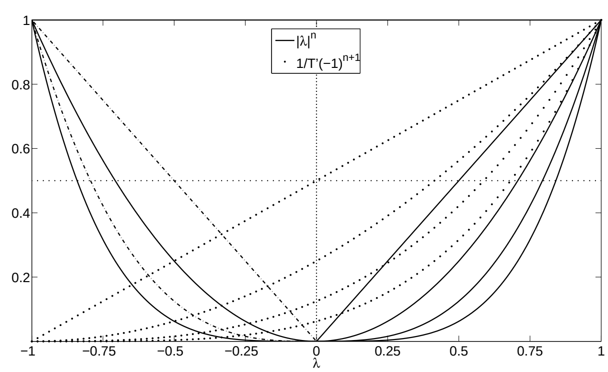

FIGURE 3. For each $\lambda \in (-1, 1)$ and for $n = 0, \ldots, 4$ the eigenvalues in the spectrum (21) of $\mathcal{L}_I$ are plotted (in modulus). These are comprised of the eigenvalues λ^n of $\mathcal{L}_{\mathbb{T}}$ (solid line for $\lambda^n > 0$, and dashed for $\lambda^n < 0$) and the eigenvalues $1/(T'(-1))^{n+1}$ of $\mathcal{L}_I$. Note that the case $\lambda = 0$ corresponds to the doubling map.

To the best of our knowledge these are the first examples of nontrivial circle and interval maps for which the entire spectrum of the associated Perron-Frobenius operators is known explicitly. Certain conjectures were previously hard to test, without examples. These might now be more accessible.

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References

[1] I. Antoniou and S. Tasaki. Spectral decomposition of the Rnyi map. *J. Phys. A* 26(1):73–94, 1993.
[2] R. Artuso, E. Aurell, and P. Cvitanovic. Recycling of strange sets: I. Cycle expansions. *Nonlinearity*, 3(2):325–359, 1990.
[3] V. Baladi. *Positive transfer operators and decay of correlations*. World Scientific Publishing Co. Pte. Ltd, Singapore, 2000.
[4] V. Baladi and M. Holschneider. Approximation of nonessential spectrum of transfer operators. *Nonlinearity*, 12(3):525–538, 1999.
[5] O. F. Bandtlow and O. Jenkinson. Invariant measures for real analytic expanding maps. *J. Lond. Math. Soc.*, 75(2):343–368, 2007.
[6] O. F. Bandtlow and O. Jenkinson. Explicit a priori bounds on transfer operator eigenvalues. *Commun. Math. Phys.*, 276(3):901–905, 2007.
[7] O. F. Bandtlow and O. Jenkinson. Explicit eigenvalue estimates for transfer operators acting on spaces of holomorphic functions. *Adv. Math.*, 218(3):902–925, 2008.
[8] O. F. Bandtlow and O. Jenkinson. On the Ruelle eigenvalue sequence. *Ergodic Theory Dynam. Systems*, 28(06):1701–1711, 2008.
[9] A. Boyarsky and P. Gora. *Laws of Chaos: Invariant Measures and Dynamical Systems in One Dimension (Probability and its Applications)*. Birkhäuser, Basel, 1997.
[10] F. Christiansen, G. Paladin, and H. H. Rugh. Determination of correlation spectra in chaotic systems. *Phys. Rev. Lett.*, 65(17):2087–2090, 1990.
[11] P. Collet and J. P. Eckmann. Liapunov Multipliers and Decay of Correlations in Dynamical Systems. *J. Stat. Phys.*, 115(1/2):217–254, 2004.
[12] J. B. Conway. *Functions of One Complex Variable*. Springer, New York, 1973.
[13] F. Diakonos and P. Schmelcher. On the construction of one-dimensional iterative maps from the invariant density: the dynamical route to the beta distribution. *Phys. Lett. A*, 211(4):199–203, 1996.
[14] N. Dunford and J. T. Schwartz. *Linear Operators, Part 2: Spectral Theory*. Wiley-Interscience, 1963.
[15] S. V. Ershov and G. G. Malinetskii. The solution of the inverse problem for the Perron-Frobenius equation. *USSR Computational Mathematics and Mathematical Physics*, 28(5):136–141, 1988.
[16] D. Fried. The zeta functions of Ruelle and Selberg.I. *Ann. Sci. Éc. Norm. Supér.*, 19(4):491–517, 1986.
[17] G. Froyland. Computer-assisted bounds for the rate of decay of correlations 1 Introduction and Motivation. *Comm. Math. Phys.*, 189(2):237–257, 1997.
[18] P. Gaspard. r-adic one-dimensional maps and the Euler summation formula. *J. Phys. A* 25(8):L483–L485, 1992.
[19] P. Gora and A. Boyarsky. A matrix solution to the inverse Perron-Frobenius problem. *Proc. Amer. Math. Soc.*, 118(2):409–414, 1993.
[20] H. H. Hasegawa and W. C. Saphir. Decaying eigenstates for simple chaotic systems. *Physics Letters A*, 161(6):471–476, 1992.
[21] O. Jenkinson and M. Pollicott. Orthonormal expansions of invariant densities for expanding maps. *Adv. Math.* 192(1):1–34, 2005.

[22] G. Keller and H. H. Rugh. Eigenfunctions for smooth expanding circle maps. *Nonlinearity*, 17(5):1723–1730, 2004.

[23] G. M. Levin. On Mayer's conjecture and zeros of entire functions. *Ergodic Theory Dynam. Systems*, 14(03):565–574, 1994.

[24] C. Liverani. Rigorous numerical investigation of the statistical properties of piecewise expanding maps. A feasibility study. *Nonlinearity* 14(3):463–490, 2001.

[25] D. H. Mayer. On the location of poles of Ruelle's zeta function. *Lett. Math. Phys.*, 14(2):105–115, 1987.

[26] D. H. Mayer. On the thermodynamic formalism for the Gauss map. *Comm. Math. Phys.*, 130(2):311–333, 1990.

[27] D. H. Mayer. Continued fractions and related transformations. In *Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces*, pages 175–222. Oxford University Press, Oxford, 1991.

[28] H. Mori, B.-C. So, T. Ose. Time-correlation functions of one-dimensional transformations. Progr. Theoret. Phys. 66(4):1266–1283, 1981.

[29] F. Naud. Entropy and decay of correlations for real analytic semi-slows. *Ann. Henri Poincaré*, 10(3):429–451, 2009.

[30] F. Naud. The Ruelle spectrum of generic transfer operators. *Discrete Contin. Dyn. Syst.*, 32(7):2521–2531, 2012.

[31] D. Ruelle. Zeta-functions for expanding maps and Anosov flows. *Invent. Math.*, 34(3):231–242, 1976.

[32] H. H. Rugh. On the asymptotic form and the reality of spectra of Perron-Frobenius operators. *Nonlinearity*, 7(3):1055–1066, 1994.

[33] J. Slipantschuk, O. F. Bandtlow, and W. Just. On the relation between Lyapunov exponents and exponential decay of correlations. *J. Phys. A.*, 46(7):075101, 2013.

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