PART III TORIC GEOMETRY (LENT 2022)
EXAMPLE SHEET 4

1. Read Section 2.5 of Fulton and complete the exercises. This leads you through a classification of smooth and proper toric surfaces.
2. Prove that every smooth proper toric surface is projective.
3. Read Section 2.6 of Fulton and prove the continued fraction expression given on page 46. This gives the self-intersection numbers of the exceptional divisors in the minimal resolution of a toric surface singularity.
4. Let Σ be a smooth and proper 2-dimensional fan. Let ρ be a ray given by the intersection of 2-dimension cones σ_1 and σ_2. Form $\Sigma^\dagger$ by blowing up $\sigma_1 \in \Sigma$ (i.e. take the star subdivision). Compare the Cartier divisor $D_\rho \subseteq X_{\Sigma^\dagger}$ to the pullback of the Cartier divisor $D_\rho \subseteq X_\Sigma$ (the former is the strict transform, the latter the total transform). How does the self intersection of D_ρ change under the blowup?
5. Let Σ be the fan of $\mathbb{P}^1 \times \mathbb{P}^1$ and let H_1, H_2 be the invariant divisors corresponding to $(1,0),(0,1)$. For strictly positive integers a, b, compute the polytopes P_D associated to the following divisors: (1) aH_1, (2) $aH_1 + bH_2$. Which are basepoint-free and which are ample? Could you also have proved this without using the toric dictionary?
6. Let $f\colon X \to Y$ be a toric morphism between proper toric varieties. Prove that the pullback $f^\star L$ of a basepoint-free line bundle is basepoint-free. Show that the pullback of an ample line bundle need not be ample.
7. Let σ be the cone over a square, i.e. the cone in $\mathbb{R}^3$ generated by $e_3, e_1+e_3, e_2+e_3, e_1+e_2+e_3$. Let D_1, D_2, D_3, D_4 be the invariant divisors corresponding to the four rays. Give necessary and sufficient conditions for the following Weil divisor to be Cartier:
$$D = a_1D_1 + a_2D_2 + a_3D_3 + a_4D_4.$$
8. Complete the exercise on page 71 of Fulton, giving an example of a smooth proper toric 3-fold which is not projective.
9. Prove that there exist smooth proper toric varieties of every dimension at least 3 which are not projective.
10. ($\star$) Let X_Σ be a smooth projective toric variety and let $X_{\Sigma'}$ be the blowup of a toric stratum. Prove that $X_{\Sigma'}$ is also projective.
11. Work out the quotient description of each Hirzebruch surface $\mathbb{F}_k$. Hence prove that $\mathbb{F}_k$ is a fibrewise compactification of the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(k)$.

12. Calculate the rational cohomology groups and the ring structure for: $\mathbb{P}^n$, $\mathbb{P}^2 \times \mathbb{P}^2$, $\mathbb{P}^2 \times \mathbb{P}^1$ and the Hirzebruch surfaces $\mathbb{F}_k$. Interpret the answers geometrically, focusing especially on the ring structure.
13. ($\star$) Let X_Σ be a toric variety and let φ be a strictly convex piecewise linear function giving rise to a projective embedding of X. Let τ be a cone of Σ. Give a combinatorial algorithm that describes the restriction of $\mathcal{O}_X(D_\varphi)$ to the closed stratum $V(\tau)$.

Just for fun.

A. ($\star$) Let X be the surface obtained from $\mathbb{P}^2$ by blowing up three collinear points. Prove that X is not a toric variety.

B. ($\star$) There exist smooth proper toric varieties that admit no non-constant morphisms to projective space. The example was first constructed in 2005 by Payne in this paper `https://arxiv.org/pdf/math/0501204.pdf`. Study and summarise this example.

C. ($\star$) (Chow Lemma) Let X_Σ be a proper toric variety. Prove that there is a toric projective variety $X_{\Sigma'}$ birational to X_Σ via a toric morphism $X_{\Sigma'} \to X_\Sigma$.

D. ($\star$) (Toric Nakai Criterion[1] for line bundles) Prove that if X is a complete toric variety and L is a line bundle, L is ample if and only if the degree of $L|_C$ is strictly positive for every irreducible boundary curve C.

[1] The Nakai criterion holds more generally and is quite useful from time to time. Look it up.