Desingularising genus 1 moduli

Two key ideas:

curves w/ elliptic singularities ① $\longleftrightarrow$ log blowups induced by tropical order relations ②

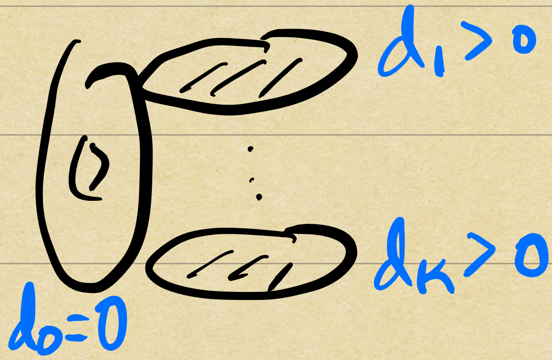
Ranganathan–Santos-Parker–Wise I ('17)
absolute target.

$\downarrow$

- RSPWII ('17) relative toric boundary
  tropical realisability.
- Battistella–NN–Ranganathan ('19)
  relative smooth divisor
  recursive description of boundary / recursive algorithm
- Bozlee ('19): higher genus curves
- Battistella–Carocci ('20): $g=2$ maps.
- NN–Ranganathan ('19): Maximal contact log GW theory ($g=0$).

1) $\overline{M}_{1,n}(\mathbb{P}^r, d)$

- $\overline{M}_{1,n}(\mathbb{P}^r, d)$ has multiple irred. components.
- Main component: closure of locus where source curve is smooth; has $\dim = \operatorname{vdim} \overline{M}_{1,n}(\mathbb{P}^r, d)$.
- Excess components of $\overline{M}_{1,n}(\mathbb{P}^r, d)$:



Excess dim $= r - k$.

- Hope: isolate contribution of main component to GW invariants (reduced GW theory).

- Problem: main component not even virtually smooth.

- Goal: resolve singularities of main component.

- Already done:
  (Li-)Vakil-Zinger
  Hu-Li-Niu

- Today: alternative perspective.

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2) Curves w/ elliptic singularities

- What causes singularities of $\overline{M}_{1,n}(\mathbb{P}^r,d)$?

- Map $\overline{M}_{1,n}(\mathbb{P}^r, d) \longrightarrow \overline{M}_{1,n}$ is <u>obstructed</u> at points where <u>$H^1(C, f^* T_{\mathbb{P}^r}) \neq 0$.</u>

- Euler sequence for $T_{\mathbb{P}^r}$:

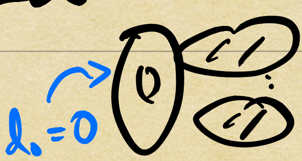
$$H^1(C, f^* T_{\mathbb{P}^r}) \neq 0 \iff H^1(C, f^* \mathcal{O}_{\mathbb{P}^r}(1)) \neq 0$$

$\iff$ the minimal genus one subcurve $E \subseteq C$ ("the <u>core</u>") is contracted by f.

normalisation sequence.

- Main cpt: generically unobstructed

Excess cpt: obstructed everywhere.



$$h^1(C, f^* T_{\mathbb{P}^r}) = r.$$

- Note: this is the same thing that causes quantum Lefschetz to fail for $g \geq 1$.

- Idea: change moduli problem, by disallowing contracted elliptic cpts.

- Need to replace by something:

$$\boxed{\text{Gorenstein curve singularities of genus one.}}$$

- Smyth '08
  Alternative modular compactification of $M_{1,n}$, allowing curves with "worse" singularities.

- Let $P \in C$ be an isolated singularity of a curve, and

$$\nu : \tilde{C} \to C$$

the normalisation at P.

Define invariants of singularity:

- $m = \# \nu^{-1}(P)$

  \# branches of C at P.

- $\delta = \dim_k \left(\nu_* \mathcal{O}_{\tilde{C}} / \mathcal{O}_C \middle|_P \right)$

  \# conditions for a f^n on $\tilde{C}$ to descend to C.

  $(\mathcal{O}_C \to \nu_* \mathcal{O}_{\tilde{C}})$

Then define:

$$\boxed{g = g(C,P) = \delta - (m-1).}$$

\# conditions to descend, except obvious ones.

<u>E.g.</u>: $C = V(xy)$ nodal curve.

$m = 2$

$\delta = 1$ $\qquad \underset{\mathcal{O}_C}{\frac{K[x,y]}{xy}} \hookrightarrow \underset{\mathcal{O}_{\tilde{C}}}{\underline{K[x] \times K[y]}}$

$\Rightarrow g = \delta - (m-1) = 0.$

Another way to say this:

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r, hook] \arrow[d, hook] & \mathbb{A}^1_y \arrow[d] \arrow[ddr, bend left] \arrow[dl, phantom, "\lrcorner"] & \\
\mathbb{A}^1_x \arrow[r] \arrow[drr, bend right] & V(xy) \arrow[dr] & \\
& & Z
\end{tikzcd}$$

<u>E.g</u>: C = union of n co-ordinate lines in $\mathbb{A}^n$
"rational n-fold point"

$m = n$, $\delta = n-1$

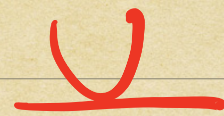
$\Rightarrow g = 0$.

Why "genus"? If smoothen and apply semistable reduction, central fibre has nodal curve of genus g in place of singularity

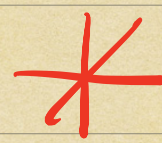
Theorem (Smyth): For each $m \geq 1$, $\exists!$ Gorenstein curve singularity with $g = 1$ and m branches:

$m = 1$: cusp 

$C = V(y^2 - x^3)$

$m = 2$: tacnode 

$C = V(y(y - x^2))$

$m = 3$: planar triple pt 

$m \geq 3$: union of m general lines in $\mathbb{A}^{m-1}$

"Elliptic m-fold point"

E.g.: cusp ($n=1$).

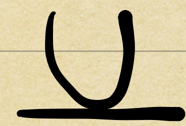
$$\mathcal{O}_C = \frac{K[x,y]}{x^3-y^2} = \underset{\underset{K[\mathbb{N}\setminus\{1\}]}{\parallel}}{K[t^2,t^3]} \subseteq \underset{\underset{\mathcal{O}_{\tilde{C}}}{\parallel}}{K[t]}$$

$\Rightarrow \delta = 1 \Rightarrow g = 1$.

- $f(t) \in K[t] = \mathcal{O}_{\tilde{C}}$ descends to C iff $\partial f/\partial t|_{t=0} = 0$

E.g.: Tacnode: glue 2 lines along a tangent vector.

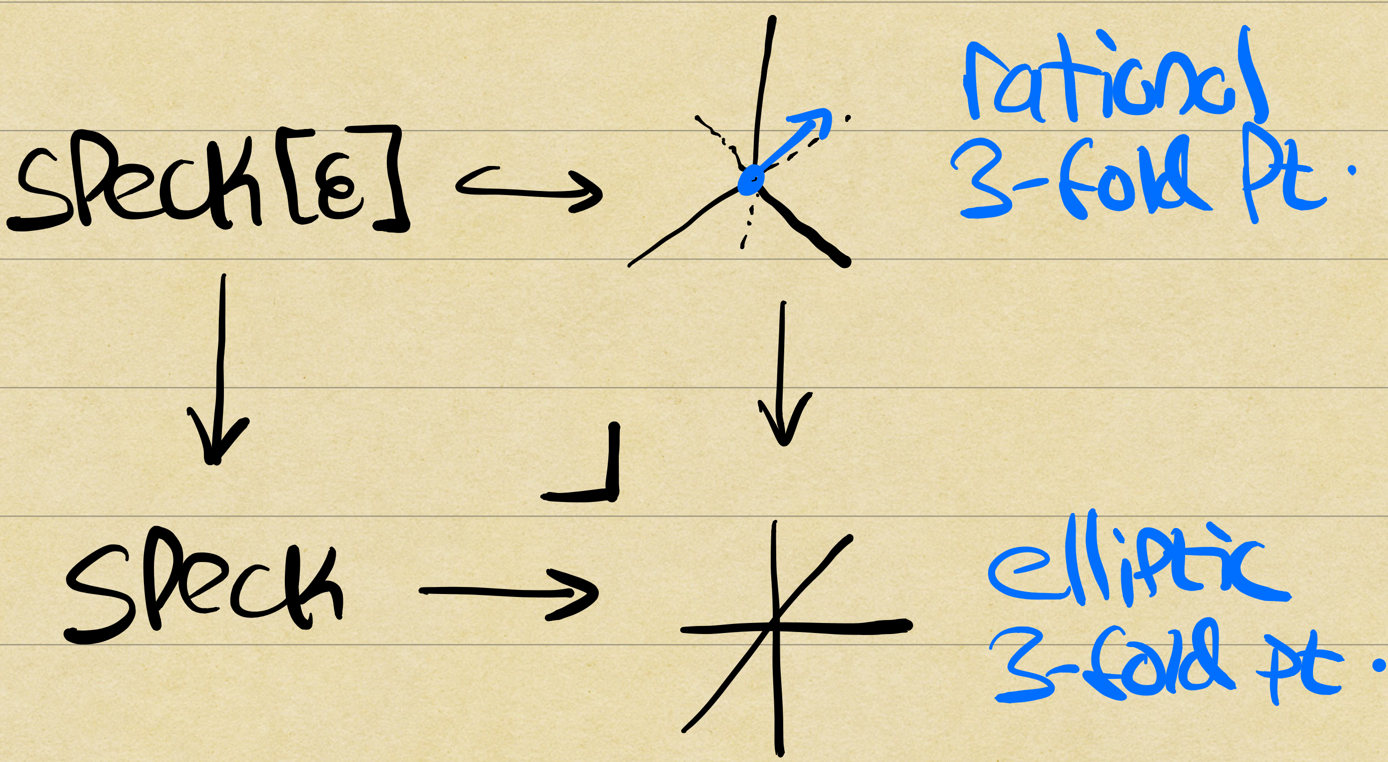
$$\begin{tikzcd}
\operatorname{Spec} k[\varepsilon] \arrow[r] \arrow[d, hook] & \mathbb{A}^1_s \arrow[d] \arrow[ddr, bend left] & \\
\mathbb{A}^1_t \arrow[r] \arrow[drr, bend right] & C \arrow[ul, phantom, very near start, "\lrcorner"] \arrow[dr] & \\
 & & Z
\end{tikzcd}$$

$C =$ 

we see extra condition to descend!

$$df_1|_0 = df_2|_0$$

E.g: Planar triple point: obtained from spatial triple point by collapsing general tangent vector.



- This is general: an elliptic m-fold pt is obtained from a rational m-fold pt by collapsing a general tangent vector.

  So see the 1 extra condition.

- Smyth uses these to give alternative modular compactifications of $M_{1,n}$.

- Viscardi ('10): Same for stable maps.

<u>Idea</u>:

① disallow contracted elliptic components.

② allow elliptic m-fold singularities.

- Kills excess components!
- Battistella-Carocci-Manolache ('18): for the quintic 3-fold, Viscardi invariants agree with Li-Zinger reduced invariants (main component contribution).

- But Viscardi's space is not a desingularisation of the main component:

1) It's <u>not smooth</u> in general.
(It's relatively unobstructed, but space of smyth curves is singular.)

2) There <u>isn't even a map</u>:
$\mathrm{Vis}_{1,n}(\mathbb{P}^r,d) \not\to \overline{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$.

- Both Viscardi's space and the main component of stable maps contain the same dense open.

But there isn't a map in either direction.

To understand why, need to understand how Smyth curves are constructed.

(This will give connection to log structures.)

- Q: How to build a Smyth curve (C, P) from its pointed normalisation $(\tilde{C}, P_1 + \dots + P_m)$?

- A: Need the extra data of the hyperplane:

$$(*) \quad \nu^* \Omega_{C,P} \subseteq \bigoplus_{i=1}^{m} \Omega_{\tilde{C}, P_i}$$

3 ways to think of this:

(i) consists of those functions on $\tilde{C}$ which descend to C.

(ii) normal vector specifies the tangent vector to collapse.

(iii) identifies tangent spaces

$$T_{\tilde{C},p_i} = T_{\tilde{C},p_j} \quad (=T)$$

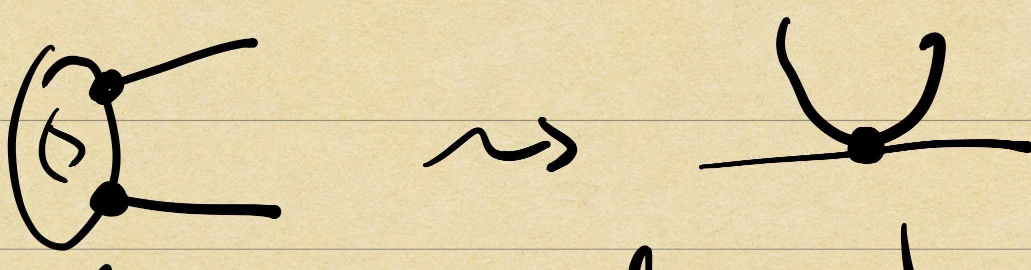
For $\tilde{f}: \tilde{C} \to Z$, gives:

$$\sum_i d\tilde{f}|_{p_i} : T \to T_{Z,\tilde{f}(q)}$$

whose vanishing is equivalent to collapse of tangent vector in (ii).

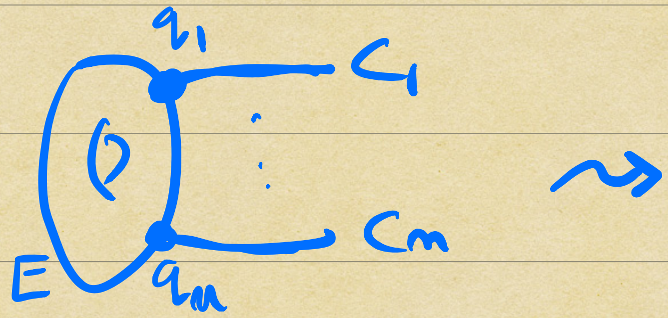
"Moduli of attachments"

- Consequence: no way to go



Need more data!

- what if we smooth?



Get vector in normal space to stratum:

$$\bigoplus_{i=1}^{m} \left(T_{q_i} E \otimes T_{q_i} C_i \right).$$

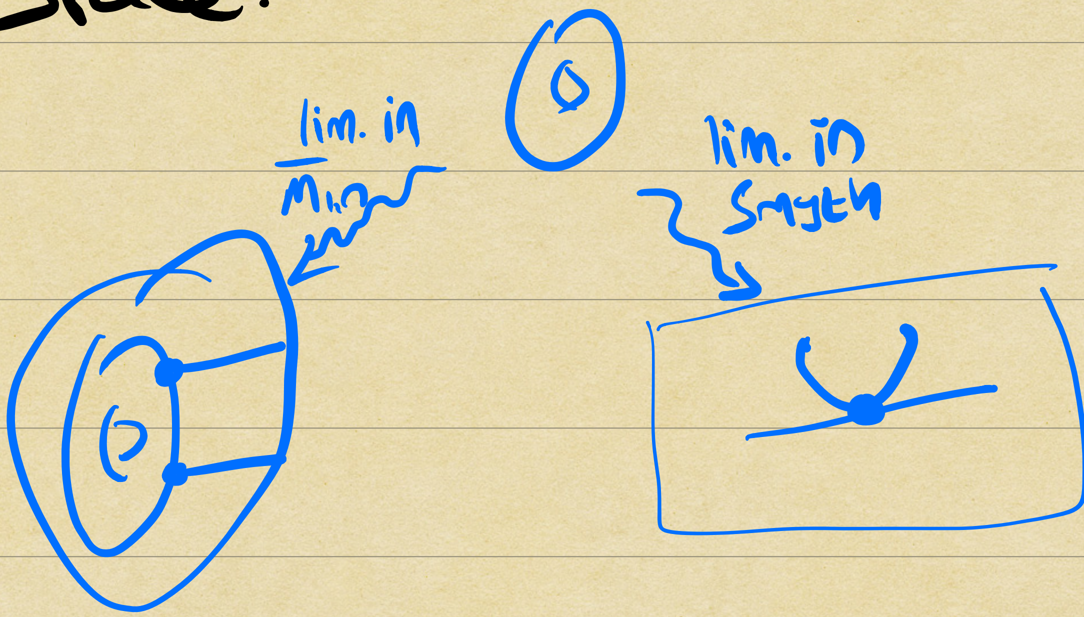
But $T_{q_i} E = T_{q_j} E$ (group law). So really get vector in

$$\bigoplus_{i=1}^{m} T_{q_i} C_i$$

I.e. hyperplane in

$$\bigoplus_{i=1}^{m} \Omega_{q_i} C_i$$

Associated elliptic singularity is central fibre in Smyth space:



- Idea: add extra data to stable maps, to encode a smoothing:

Log structures!

3) Radial alignments (RSPW)

1. Big Picture: blowup $\overline{M}_{1,n}(\mathbb{P}^r,d)$

$$\widetilde{VZ}_{1,n}(\mathbb{P}^r,d) \to \overline{M}_{1,n}(\mathbb{P}^r,d)$$

so that exceptional directions contain smoothing information.

Get contraction of universal curve:

$$\mathcal{C} \to \overline{\mathcal{C}}$$

Smyth curve

2. Pass to closed substack where map to $\mathbb{P}^r$ factors through $\overline{\mathcal{C}}$:

$$\begin{tikzcd} C \arrow[r] \arrow[dr, "f"'] & \overline{C} \arrow[d, dashed, "\overline{f}"] \\ & \mathbb{P}^r \end{tikzcd}$$

$\Rightarrow$ Obstruction theory given by $H^i(\overline{C}, \overline{f}^* T_{\mathbb{P}^r})$.

Have:

$$\begin{tikzcd} VZ_{1,n}(\mathbb{P}^r, d) \arrow[r, hook] \arrow[d] & \widetilde{VZ}_{1,n}(\mathbb{P}^r, d) \arrow[d] \\ \overline{M}^{main}_{1,n}(\mathbb{P}^r, d) \arrow[r, hook] & \overline{M}_{1,n}(\mathbb{P}^r, d) \end{tikzcd}$$

Then $VZ_{1,n}(\mathbb{P}^r,d)$ is a desingularisation of the main component.

$$\begin{tikzcd}
 & VZ_{1,n}(\mathbb{P}^r,d) \arrow[dl] \arrow[dr] & \\
Vis_{1,n}(\mathbb{P}^r,d) \arrow[rr, dashed, leftrightarrow] & & \overline{M}^{main}_{1,n}(\mathbb{P}^r,d)
\end{tikzcd}$$

.

- $\overline{M}_{1,n}(\mathbb{P}^r,d)$ has a log structure coming from its virtual boundary

- Gives tropicalisation :

$$T_{1,n}(\mathbb{P}^r,d) = \mathrm{Trop}\, \overline{M}_{1,n}(\mathbb{P}^r,d).$$

- What you need to know about $T_{1,n}(\mathbb{P}^r, d)$:

1) Plays role of "fan" of $\overline{M}_{1,n}(\mathbb{P}^r, d)$.

   - Is an abstract cone complex.
   - orbit-cone correspondence:

$$\left\{ \begin{matrix} \text{cones in} \\ T_{1,n}(\mathbb{P}^r, d) \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{boundary} \\ \text{strata in} \\ \overline{M}_{1,n}(\mathbb{P}^r, d) \end{matrix} \right\}.$$

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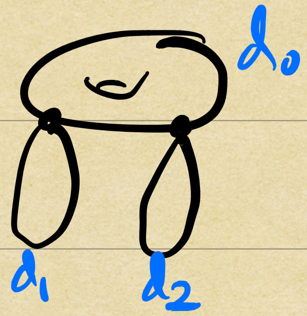
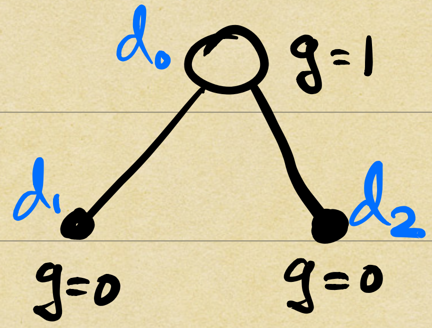
2) Can be thought of as a moduli space of tropical curves.

E.g.: Consider stratum in $\overline{M}_{1,n}(\mathbb{P}^r, d)$:



Associate dual graph:

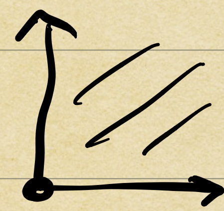


Then corresponding cone

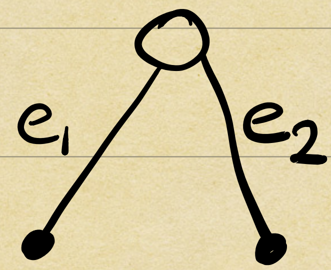
$$\sigma \in T_{1,n}(\mathbb{P}^r, d)$$

is:

$$\sigma \cong \mathbb{R}^2_{\geq 0} =$$



Think of as moduli for edge lengths:

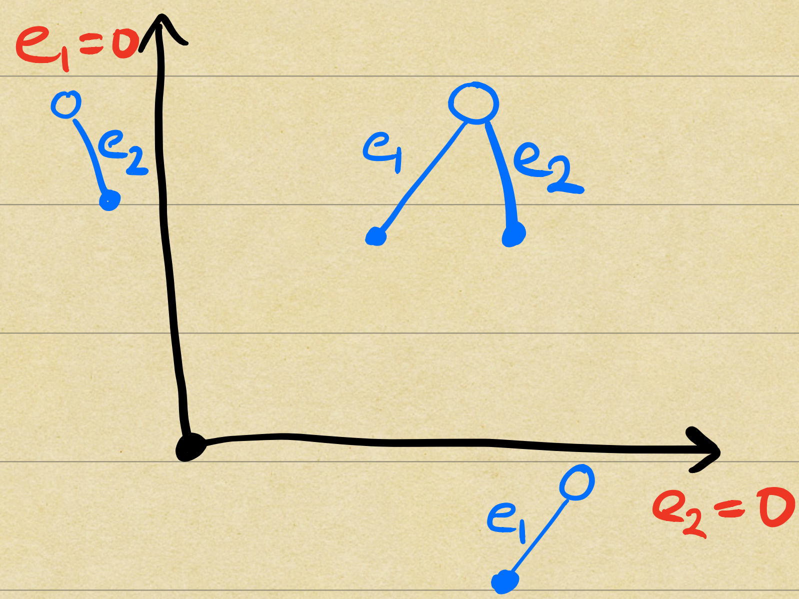


$$\sigma = (\mathbb{R}^2_{\geq 0})_{e_1 e_2}$$

Edge lengths encode "order of Smoothing" of node. Normal directions to Stratum:

$\forall \operatorname{codim} Z(\sigma) = \dim \sigma$

Setting an edge length to 0 generises the curve, moving to a different stratum:



- Setting $e_i = 0 \leftrightarrow$ smoothing node q_i.
- face inclusions dual to strata inclusions.

3) Two key constructions from toric geometry carry over:

(i) subdivision of $T_{1,n}(\mathbb{P}^r, d)$ $\rightsquigarrow$ birational modification of $\overline{M}_{1,n}(\mathbb{P}^r, d)$

(ii) PL function on $T_{1,n}(\mathbb{P}^r, d)$ $\rightsquigarrow$ Cartier divisor on $\overline{M}_{1,n}(\mathbb{P}^r, d)$.

- we'll use (i) to build $\widetilde{VZ}_{1,n}(\mathbb{P}^r, d) \to \overline{M}_{1,n}(\mathbb{P}^r, d)$.

- Idea: define another tropical moduli space, with map

$$\widetilde{T}_{1,n}(\mathbb{P}^1,d) \to T_{1,n}(\mathbb{P}^1,d).$$

which is a subdivision.

- Let's make precise statement that $T_{1,n}(\mathbb{P}^1,d)$ is moduli space of tropical curves.

- Fix τ a cone. Then a family of tropical curves over τ consists of:

(i) a graph Γ (w/ genus, degree, marking labels).

(ii) a map $\boxed{E(\Gamma) \xrightarrow{\ell} \tau^\vee}$

- Picture: have family of curves:

$$\tilde{\Sigma} \xrightarrow{\pi} \tau$$

given $p \in \tau$, $\ell(e)(p) \in \mathbb{R}_{\geq 0}$ is edge length of $e \in \pi^{-1}(p)$.

- This defines functor:

$$(\text{cones}) \xrightarrow{F} (\text{sets})$$

$$\tau \mapsto \left\{ \begin{matrix} \text{families of} \\ \text{tropical} \\ \text{curves over} \\ \tau \end{matrix} \right\} \Big/ \sim$$

And we have:

$$(\ast) \quad F(-) = \mathrm{Hom}_{\text{cones}}(-, T_{1,n}(\mathbb{P}^r, d))$$

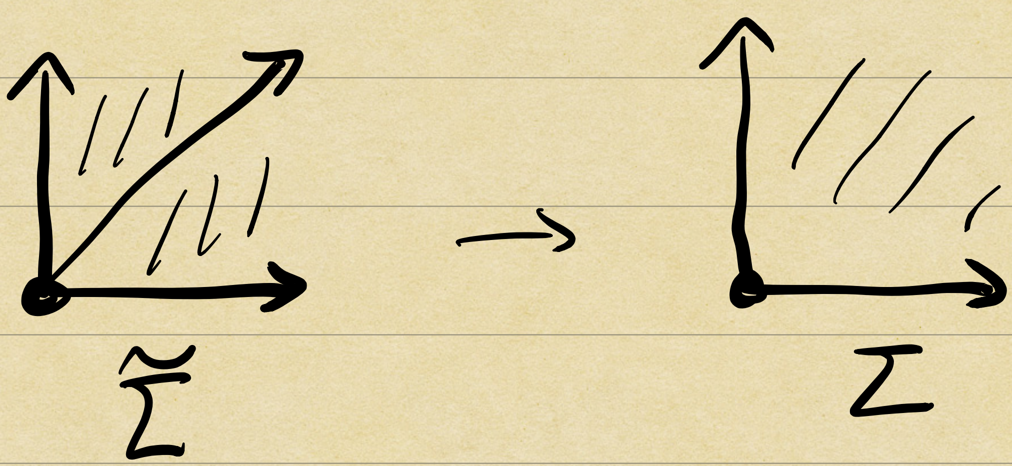
(Really should be a CFG; cf. Cavalieri-Chan-Ulirsch-Wise.)

- observation: if $\tilde{\Sigma} \to \Sigma$ is a subdivision, then

$$\mathrm{Hom}_{\text{cones}}(-, \tilde{\Sigma}) \subseteq \mathrm{Hom}_{\text{cones}}(-, \Sigma)$$

subfunctor.

E.g.:

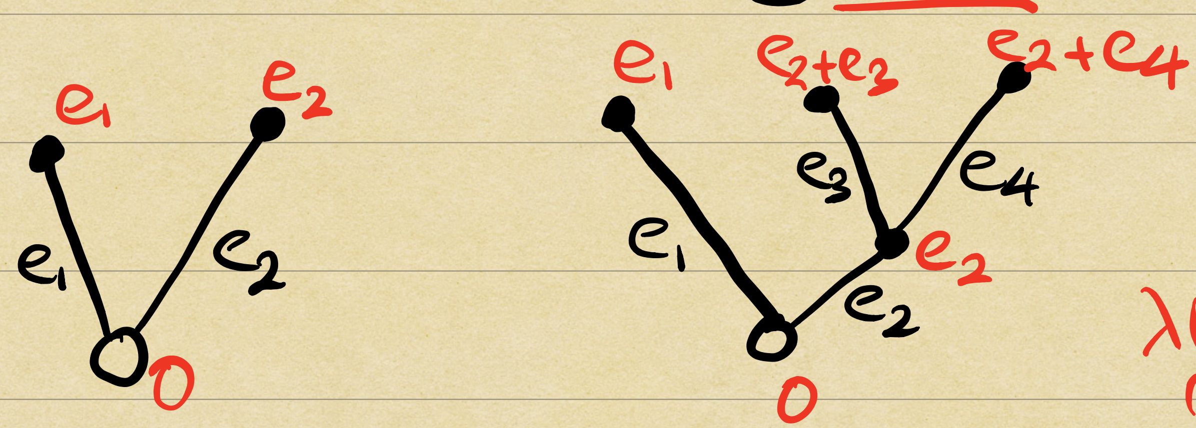


To construct Smyth curve, want to identify tangent spaces at nodes adjacent to $g=1$ subcurve to be contracted.

- **Defn:** Γ a tropical curve over τ. For $v \in V(\Gamma)$, let

$$\lambda(v) \in \tau^{\vee} \quad (*)$$

be the sum of edge lengths connecting v to minimal $g=1$ subcurve ("core").



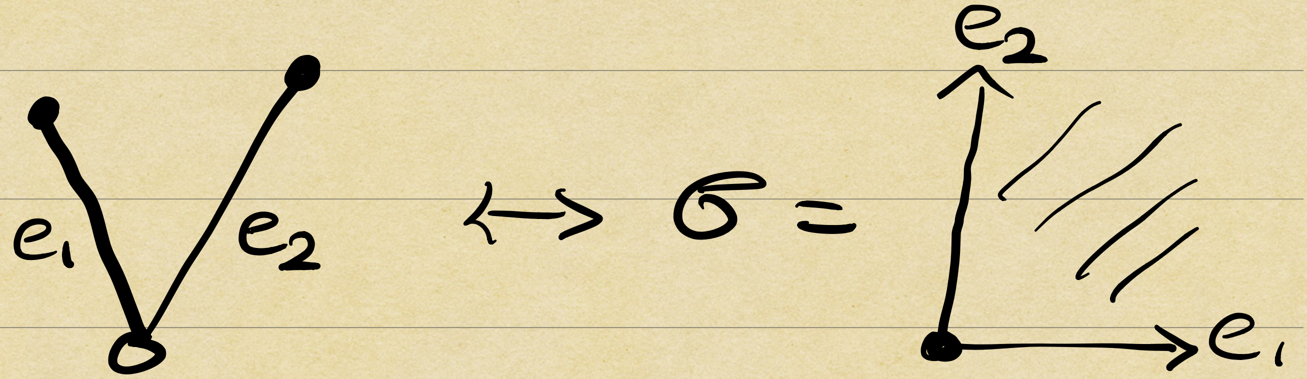
$\lambda(v)$ in red.

- Then Γ is radially aligned iff the $\lambda(v)$ are totally ordered.

$$\lambda(v) \geq \lambda(w) \iff \lambda(v) - \lambda(w) \in \tau^\vee.$$

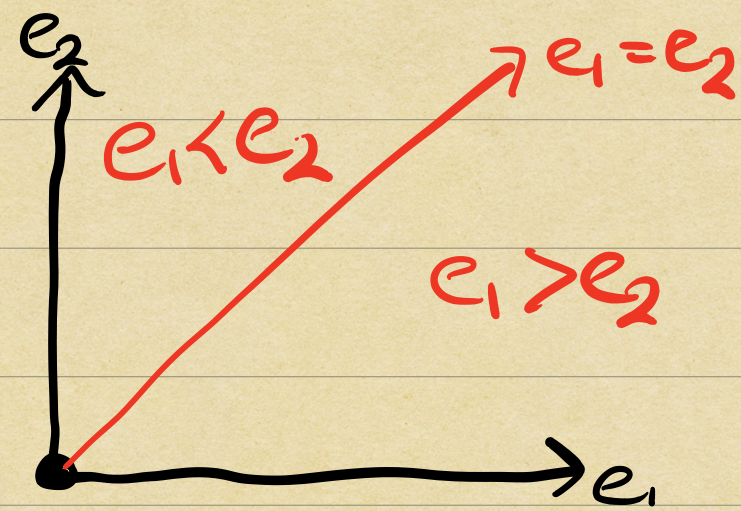
- Let $\widetilde{T}_{1,n}(\mathbb{P}^r, d)$ be moduli space of radially aligned curves. clearly subfunctor of $T_{1,n}(\mathbb{P}^r, d)$.

- How is this a subdivision of $T_{1,n}(\mathbb{P}^r, d)$?

E.g.: 

Not radially aligned: e_1 and e_2 not comparable on σ.

- Subdivide σ into regions:

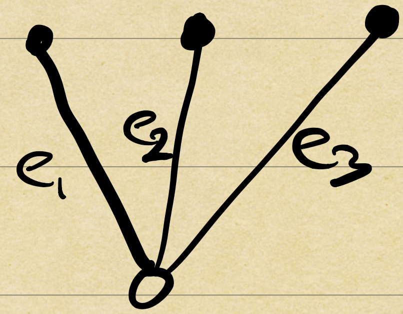


Now comparable! This describes

$$\widetilde{T}_{1,n}(\mathbb{P}^1, d) \to T_{1,n}(\mathbb{P}^1, d)$$

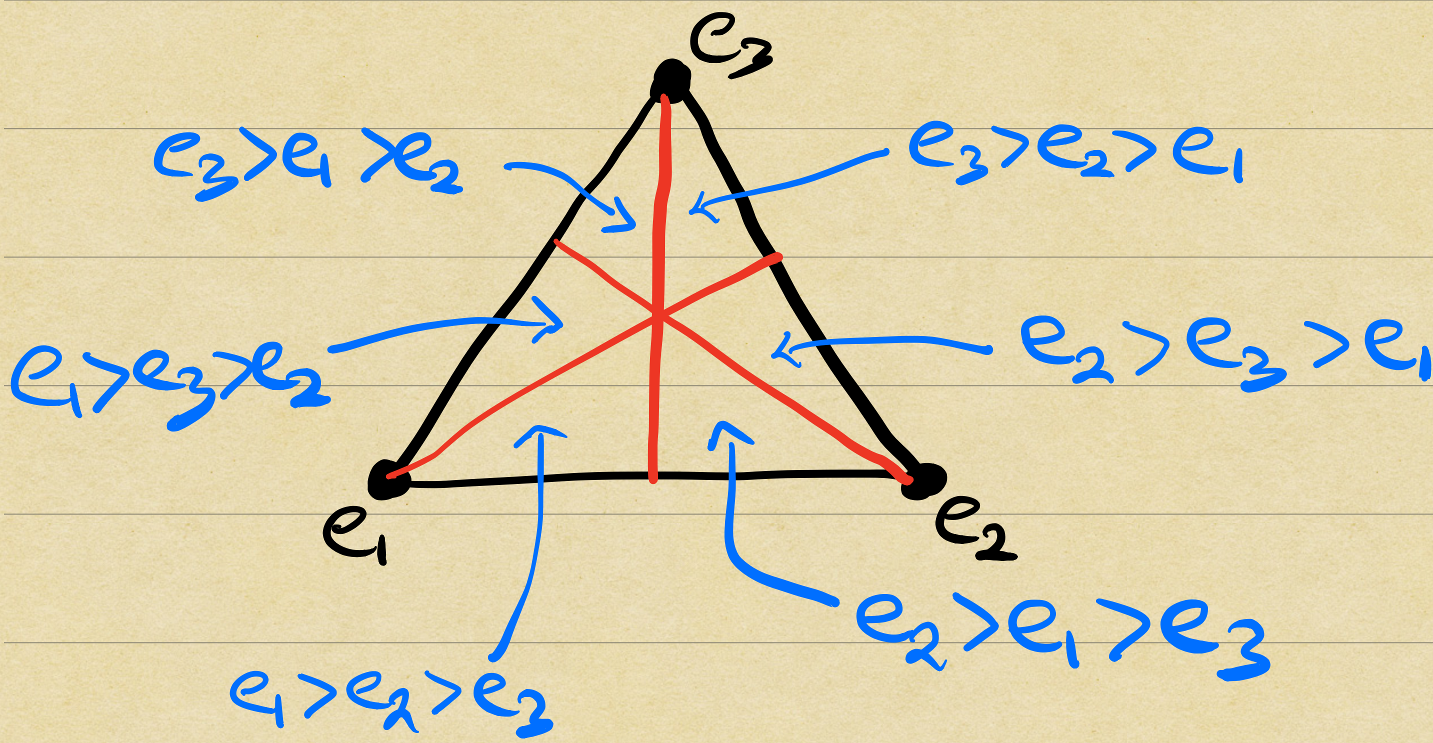
over the cone σ.

E.g.:



$\sigma \cong \mathbb{R}^3_{\geq 0}$,

Height-1 slice of subdivision:



- Defines "birational" modification

$$\widetilde{Vz}_{1,n}(\mathbb{P}^r, d) \longrightarrow \overline{M}_{1,n}(\mathbb{P}^r, d).$$

- ⚠ Not really birational, b/c some strata of $\overline{M}_{1,n}(\mathbb{P}^r,d)$ have wrong codim.

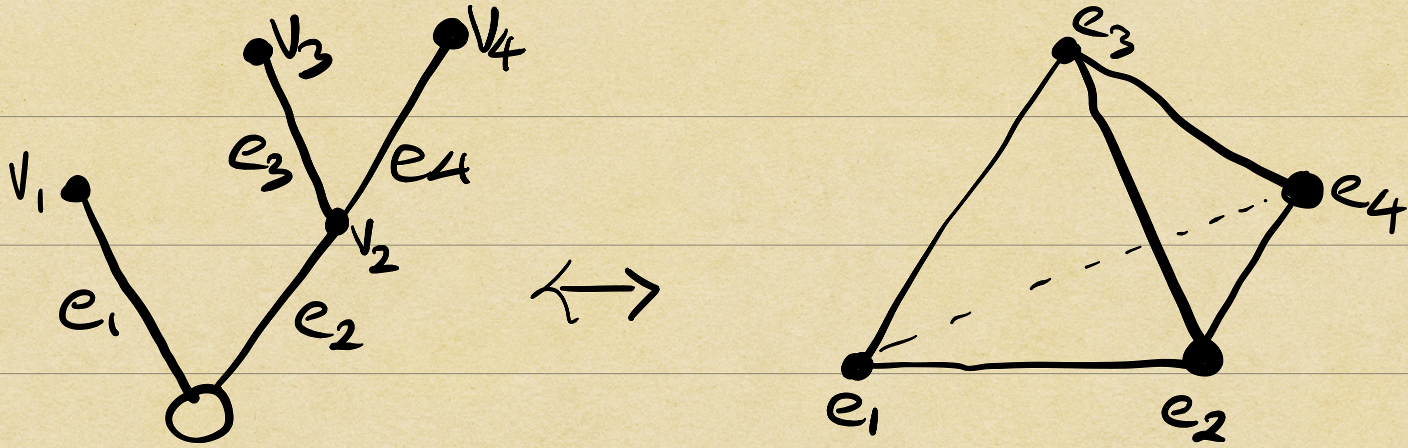
- Better to think of as pullback:

$$\begin{tikzcd}
\widetilde{VZ}_{1,n}(\mathbb{P}^r,d) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{1,n}(\mathbb{P}^r,d) \arrow[d] \\
\widetilde{\mathfrak{M}}^{wt}_{1,n} \arrow[r] & \mathfrak{M}^{wt}_{1,n}
\end{tikzcd}$$

Related to, but different from, Hu-Li blowup.

- How to describe as an iterated blowup?

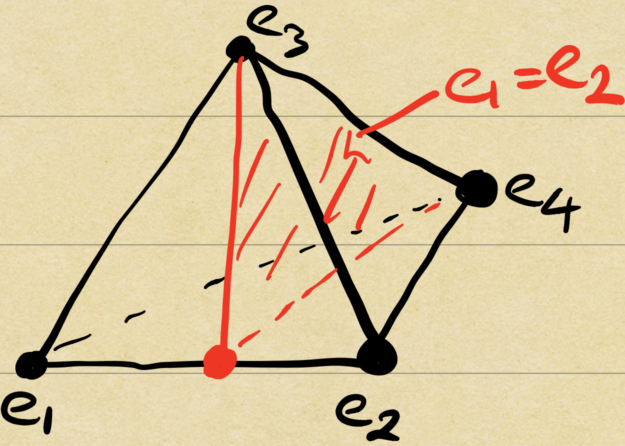
Pure combinatorics.



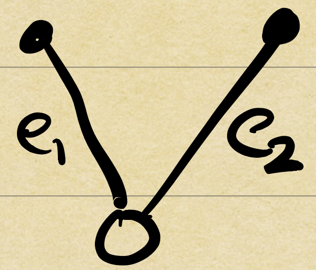
$\sigma \in T_{1,n}(\hat{\mathbb{P}}_{,d})$

- First step: Specify which $\lambda(v)$ is minimal.

2 candidates: $\left.\begin{array}{l} \lambda(v_1) = e_1 \\ \lambda(v_2) = e_2 \end{array}\right\}.$



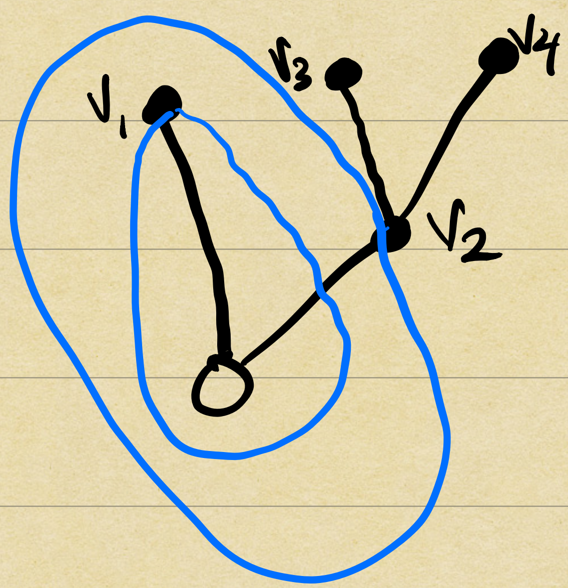
corresponds to blowup of stratum: (star subdivision)



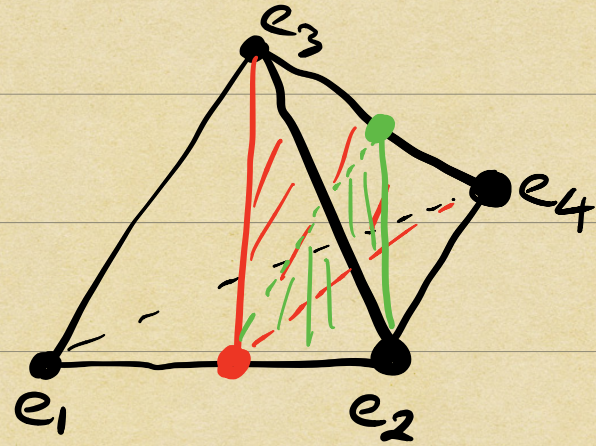
$(e_3 = e_4 = 0)$

- second step: decide which $\lambda(v)$ is next smallest.

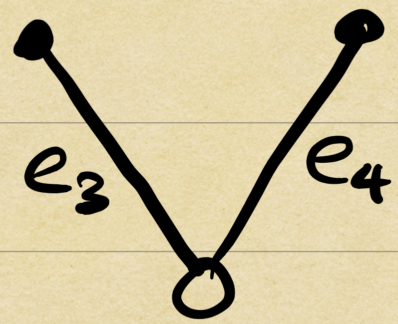
  Eg: on cone $\lambda(v_1) < \lambda(v_2)$, must have $\lambda(v_2)$ next smallest



- continue in this way. Next have to order e_3 and e_4:

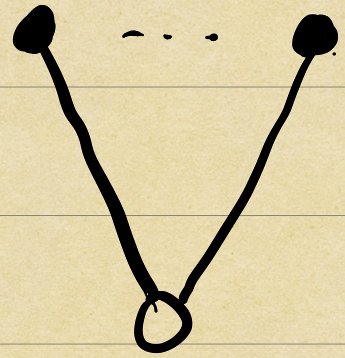


Amounts to blowing up stratum:



$(e_1 = e_2 = 0)$

- At each step, blowup stratum of form:



"flower-petal locus"

Stability ensures process respects a certain partial ordering.

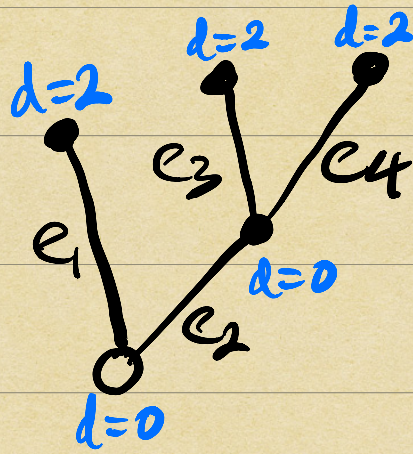
- Upshot:

  $\overline{VZ}_{1,n}(\mathbb{P}^r,d) \to \overline{M}_{1,n}(\mathbb{P}^r,d)$

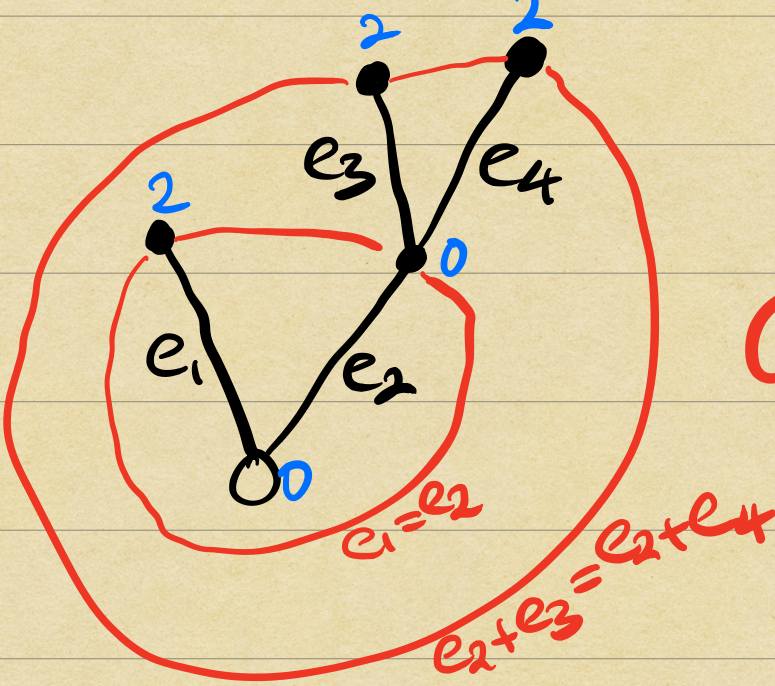
  explicit iterated blowup.

- Now connect back to Smyth curves story.

• E.g.: Consider the stratum in $\overline{M}_{1,n}(\mathbb{P}^r,d)$.



Consider stratum



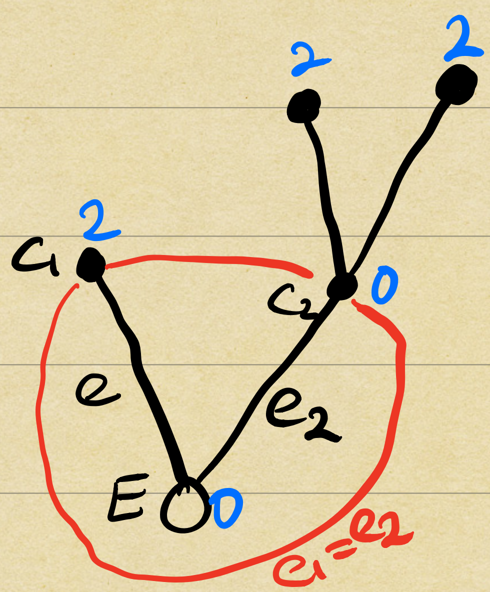
$\mathcal{O}(e_1) \cong \mathcal{O}(e_2)$

$\mathcal{O}(e_2+e_3) \cong \mathcal{O}(e_2+e_4)$.

lying over in $\widetilde{VZ}_{1,n}(\mathbb{P}^r,d)$.

• Has 2 fibre dimensions: identifications of tangent spaces

- Let $\delta =$ Minimum $\lambda(v)$ where v is a vertex of deg > 0.
- Here $\delta = e_1 = e_2$.
- δ is PL function on $\widetilde{T}_{1,n}(\mathbb{P}^r, d)$, so defines line bundle $\mathcal{O}(\delta)$.



- $\mathcal{O}(\delta) = T_{q_1}C_1 \otimes T_{q_1}E$
$$= T_{q_2}C_2 \otimes T_{q_2}E.$$

- But $T_{q_1}E = T_{q_2}E$, so get identification:

$$T_{q_1}C_1 = T_{q_2}C_2$$

as promised.

- Produces contraction to a Smyth curve:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr] \arrow[dr] & & \overline{\mathcal{C}} \arrow[dl] \\
& \widetilde{VZ}_{1,n}(\mathbb{P}^r,d) &
\end{tikzcd}$$

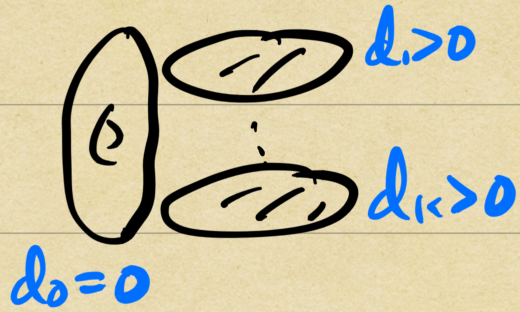
- Define $VZ_{1,n}(\mathbb{P}^r,d) \subseteq \widetilde{VZ}_{1,n}(\mathbb{P}^r,d)$ as locus where f factors:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d] & \mathbb{P}^r \\
\overline{\mathcal{C}} \arrow[ur, dashed, "\overline{f}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \overline{M}^{rad}_{1,n} \arrow[dl] \arrow[dr, "\pi"] & \\
\overline{M}_{1,n} \arrow[rr, dashed, leftrightarrow] & & \overline{M}_{1,n}(\underline{m})
\end{tikzcd}$$

(f unique if exists.)

- $VZ_{1,n}(\mathbb{P}^r, d) \to \widetilde{M}^{wt}_{1,n}$ is smooth: $H^1(E, f^*\mathcal{O}_{\mathbb{P}^r}(1)) = 0$.

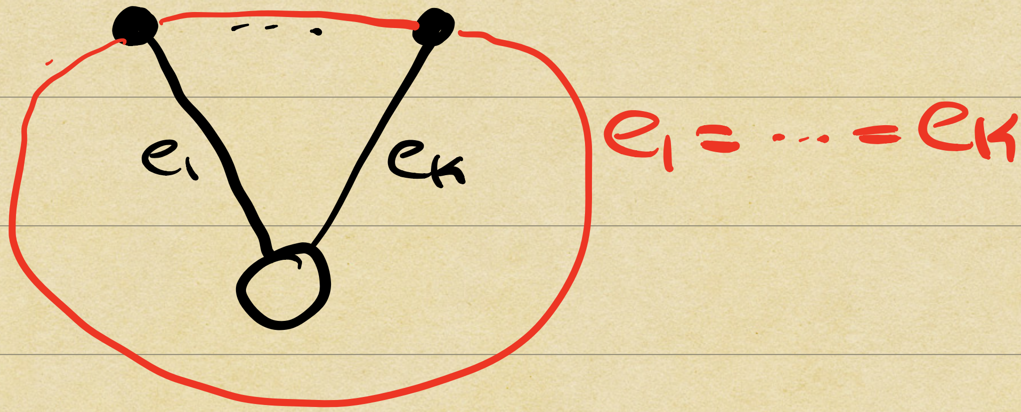
- what happened to excess cpts? Had:



Excess $\dim = r - k$

in $\overline{M}_{1,n}(\mathbb{P}^r, d)$.

- Preimage in $\widetilde{VZ}_{1,n}(\mathbb{P}^r, d)$ has maximal stratum:



Gain $k-1$ dims.

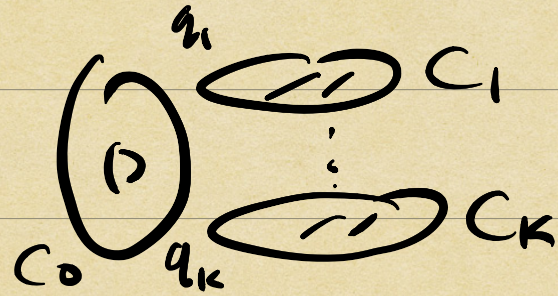
$\Rightarrow$ excess dim $= r-1$.

- Now impose factorisation: equivalent to vanishing of:

$$\sum_{i=1}^{k} df_i : \mathcal{O}(S) \to ev^* T_{\mathbb{P}^r}$$

cuts down by r dims.
$\Rightarrow$ end up w/ locus of codim. 1 in $VZ_{1,n}(\mathbb{P}^r, d)$.

- Vakil's criterion: in $\overline{M}_{1,n}(\mathbb{P}^r,d)$, a map of the form:



is in main component only if (iff?):

$$df_1(T_{C_1,q_1}),\dots, df_k(T_{C_k,q_k}) \subseteq T_{\mathbb{P}^r, f(C_0)}$$

is linearly dependent.

- Factorisation says that a specific linear dependence holds (the one specified by the identifications $T_{c_i, q_i} = T_{c_j, q_j}$).

Thank you!