Enumerative invariants of cDV resolutions

(joint w/ Michael Wemyss.)

0) du Val Singularities

$G \leq SU(2,\mathbb{C})$
finite

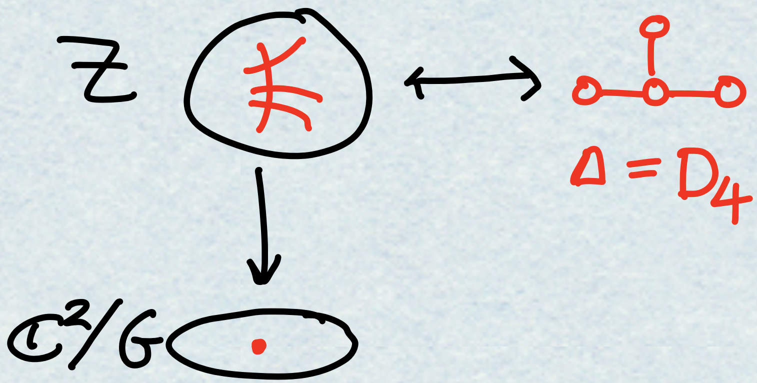
Dynkin diag. Δ	A_n $n \geq 1$	D_n $n \geq 4$	E_6	E_7	E_8
Group G	cyclic $\mathbb{Z}/n+1$	binary dihedral	— Platonic solids —		

$\mathbb{C}^2/G$ du Val Singularity.

Minimal resolution: $Z \to \mathbb{C}^2/G$.

McKay Correspondence:

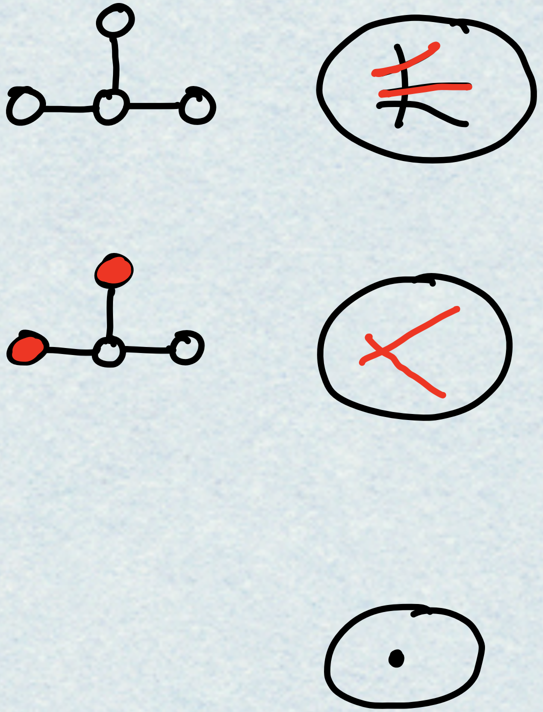
exceptional curves in Z $\longleftrightarrow$ nodes in Δ.



1) cDV Singularities.

Affine 3-fold Spec R containing $\mathbb{C}^2/G$ as hypersurface.

Assume: isolated, admits crepant resolution.



$$\begin{tikzcd} Z \arrow[d] & \\ Y \arrow[r] \arrow[d, "\text{partial res.}"'] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ \mathbb{C}^2/G \arrow[r, hook] & \operatorname{Spec} R \end{tikzcd}$$

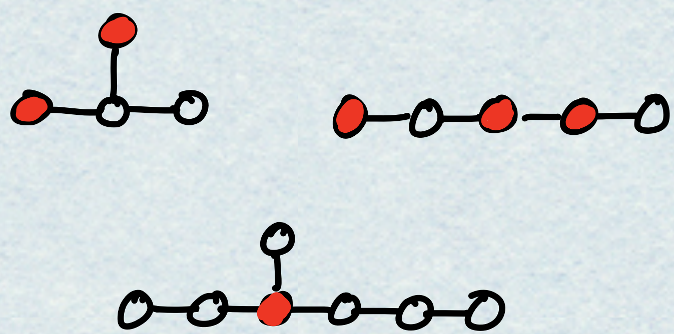
local model of 3-fold flopping contraction.

$$\boxed{A_1(X) = A_1(Y) = \bigoplus_{i \in I} \mathbb{Z}[C_i]}$$

Dynkin data: (Δ, I)

Dynkin diagram

subset of the nodes



2) Enumerative invariants

$$A_1(X) = \bigoplus_{i \in I} \mathbb{Z}[C_i] \ni \beta$$

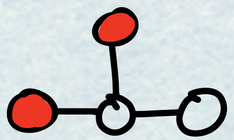
Count curves in (Perturbation of) X of class β.

- Gopakumar-Vafa: $n_\beta \in \mathbb{Z}$. (curve = sheaf.)
  [Katz]
- Gromov-Witten: $N_\beta \in \mathbb{Q}$. (curve = map.)
  [Many people]

Give local contributions to invariants of CY3s.

3) Results

Meta-th^m [N-Wemyss]: The Dynkin data (Δ, I) determines the qualitative structure of the enumerative inv'ts.



(Δ, I)

$$\rightsquigarrow \quad \begin{array}{ccc} \mathbb{Z}^{\Delta} & \twoheadrightarrow & \mathbb{Z}^{I} = A_1(X) \ni \beta \\ \| & & \| \\ \mathbb{Z}^4 & & \mathbb{Z}^2 \end{array}$$

Th^m: $n_\beta \neq 0$ iff β is a restricted root.

(Image of positive root in $\mathbb{Z}^{\Delta}$.)

Pf: Construct explicit perturbation of X.

Combinatorial from (Δ, I)

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ \mathbb{C}^2/G \arrow[r, hook] & \operatorname{Spec} R \end{tikzcd}$$

X = total space of 1-parameter deformation of Y.

= map μ : (Disk) $\longrightarrow$ (versal def. space of Y).

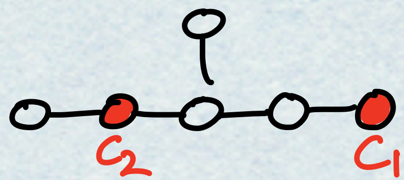
Perturb X by perturbing μ.

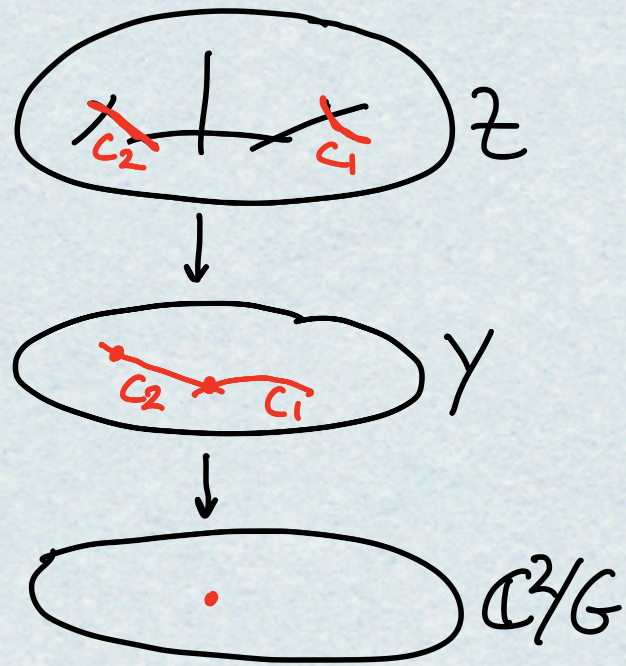
[Brieskorn, Reid, Pinkham, Katz-Morrison, Bryan-Katz-Leung.]

New input: Complete description of discriminant locus. $\square$

Th^m: $n_\beta \neq 0$ iff β is a restricted root.
(Image of positive root in $\mathbb{Z}^\Delta$.)

Cor: Pole locus of quantum product = restriction arrangement of (Δ, I).

E.g: $(\Delta, I) =$ 



Restricted roots: $\mathbb{Z}^6 \twoheadrightarrow \mathbb{Z}^2$.

$\boxed{\beta = (0,1),\ (0,2),\ (1,0),\ (1,1),\ (1,2)}$

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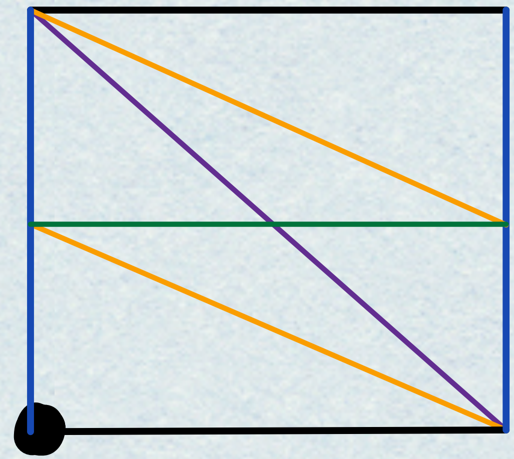
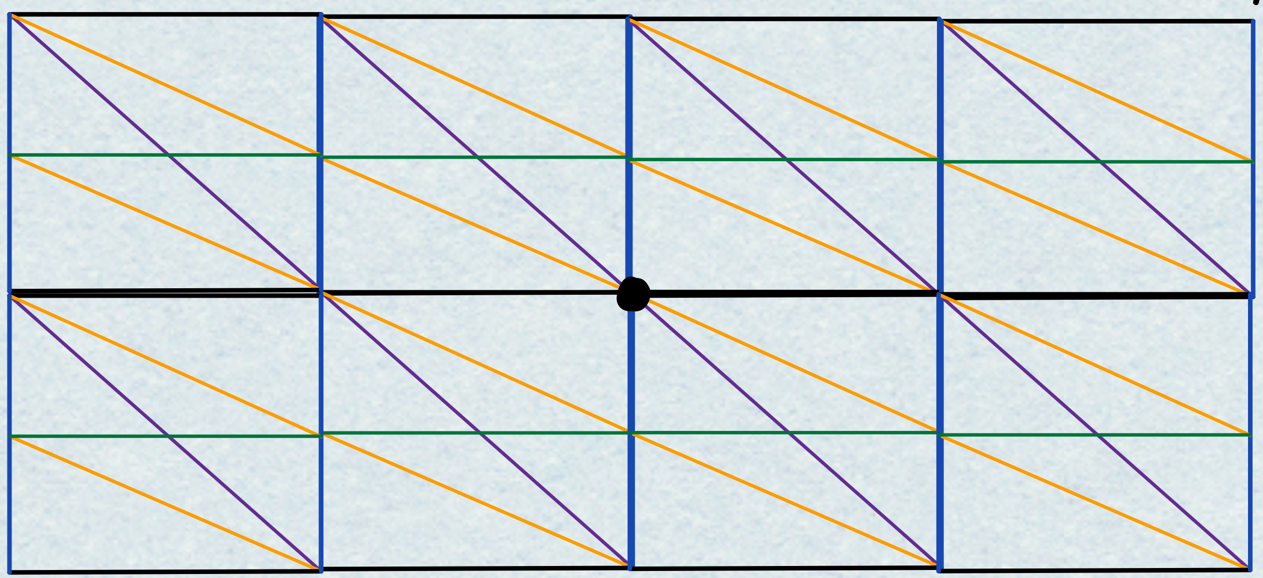


Restriction arrangement:

$H_\beta = \{ w \in \mathbb{Z}^2 : w \cdot \beta \in \mathbb{Z} \}$.

Records which $n_\beta \neq 0$.



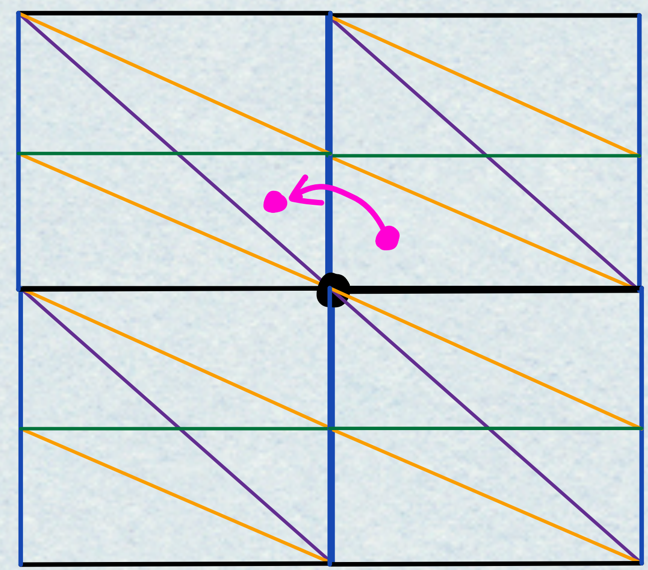
Hirano-Wemyss: base of Bridgeland stability cover.

[Bridgeland, Ikeda-Qiu, McAuley, ...]

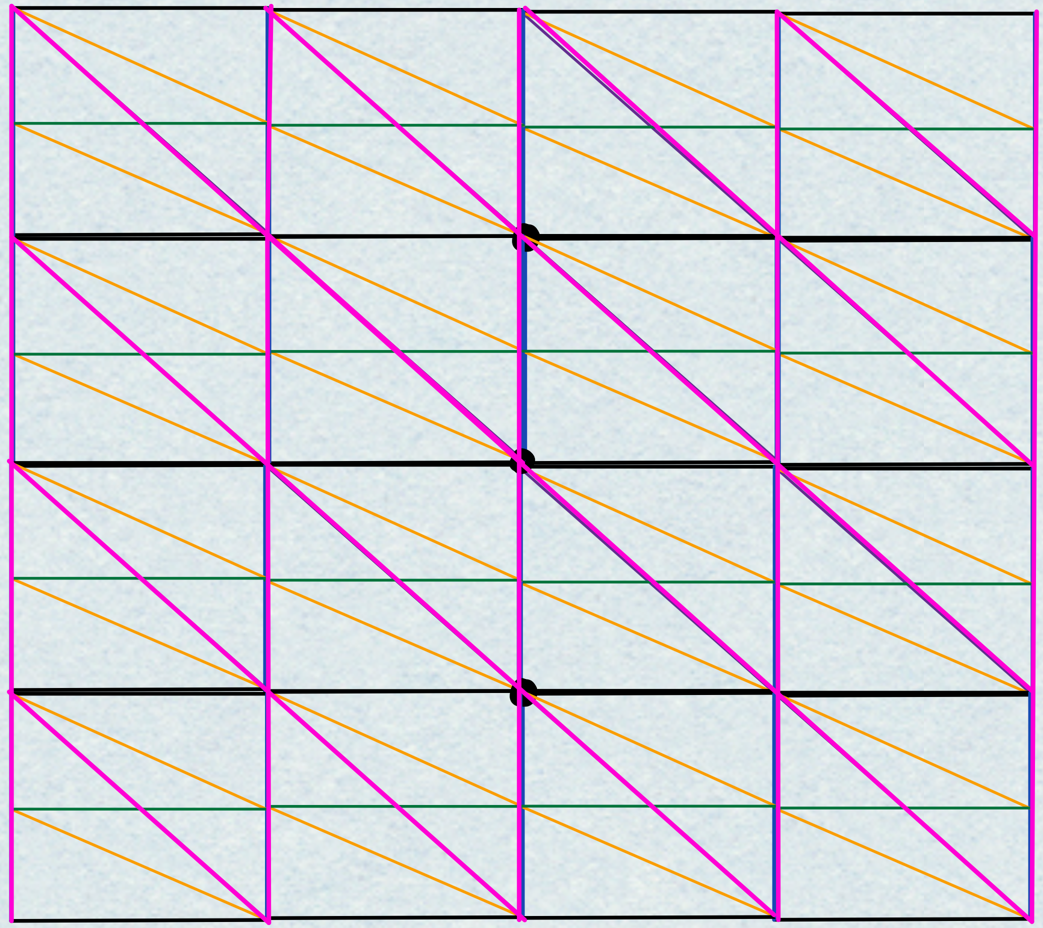
SpecR admits multiple crepant resolutions.
Related by flopping exceptional curves.

$$\begin{tikzcd} X_1 \arrow[rr, dashed, leftrightarrow] \arrow[dr] & & X_2 \arrow[dl] \\ & \operatorname{Spec} R & \end{tikzcd}$$

Wall-crossing in restriction arrangement:



Same arrangement: new fundamental domain.



Th^m: Respects invariants.

Set of Ω_β Same: reindexed by matrix.

Ω_β not combinatorial: transformation is.

Visual instance of Crepant resolution Conjecture.