Logs, roots, negative tangencies.

[Battistella-N-Ranganathan, '24]

---

1) Two theories

- Fix: $(X \mid D = D_1 + \dots + D_k)$ snc Pair

- Study: Curves in X with Specified tangency orders along D_j.

- $(C \mid P_1 + \dots + P_n) \xrightarrow{f} (X \mid D_1 + \dots + D_k)$.

  Fix $\boxed{\alpha_{ij} \in \mathbb{N}}$: tangency to D_j at P_i

- $\left\{ \begin{matrix} C \xrightarrow{f} X \\ P_1, \dots, P_n \in C \end{matrix} \;\middle|\; \forall j,\ f^{*} S_{D_j} = \prod_{i=1}^{n} S_{P_i}^{\alpha_{ij}} \right\}$

  Not compact

Need for: intersection theory/ enumerative geometry.

- Compactification: extra structure on C and X.

(a) Log Gromov-Witten theory

- Log structure: data of monomial functions

"Sheaf of monoids"

- On C: $\prod_{i=1}^{n} s_{p_i}^{c_i}$

on X: $\prod_{j=1}^{k} s_{D_j}^{d_j}$

$f^{x} s_{D_j} = \prod_{i=1}^{n} s_{p_i}^{\alpha_{ij}}$: Monomials pull back to monomials

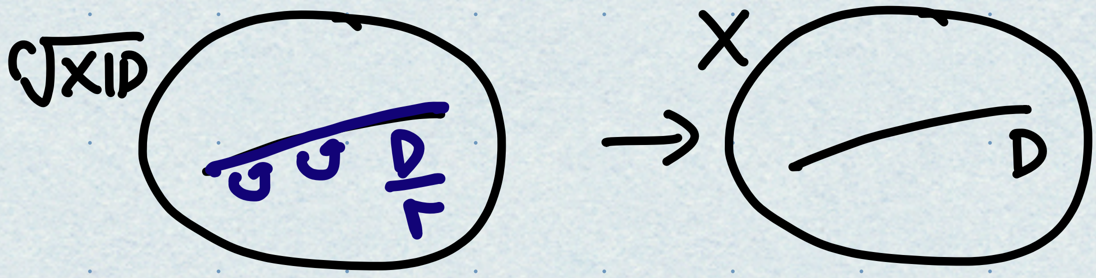
- Tangency encoded in pullback of sheaves of monoids.
- Deformation open
- Log stable maps $\rightsquigarrow$ Log GW invariants
  Log$(X|D)$

[Abramovich-Chen, Gross-Siebert '10-'11]

(b) Orbifold Gromov-Witten theory

- Fix $r \in \mathbb{N}$ (auxiliary parameter: large)
- Root stack: $\sqrt[r]{X|D} \to X$

  Universal "thing" where pullback of S_D has an rth root.



- Local model $(C|P_i) \to (X|D_j)$:

$$
\begin{tikzcd}
\mu_r \arrow[rr, "\zeta_r \mapsto \zeta_r^{\alpha_{ij}}"', "(\text{if } r > \alpha_{ij})"'] \arrow[d, phantom, "\circlearrowright"] & & \mu_r \arrow[d, phantom, "\circlearrowright"] \\
\mathbb{A}^1_{\tilde{s}} \arrow[rr, "\tilde{s}^{\alpha_{ij}} \mapsfrom \tilde{t}"'] \arrow[d] \arrow[ddd, bend right=60, "s \mapsto \tilde{s}^r"'] & & \mathbb{A}^1_{\tilde{t}} \arrow[d] \arrow[ddd, bend left=60, "t \mapsto \tilde{t}^r"] \\
{[\mathbb{A}^1_{\tilde{s}}/\mu_r]} \arrow[rr] \arrow[dd] & & {[\mathbb{A}^1_{\tilde{t}}/\mu_r]} \arrow[dd] \\
& & \\
\mathbb{A}^1_s \arrow[rr, "f", "s^{\alpha_{ij}} \mapsfrom t"'] & & \mathbb{A}^1_t
\end{tikzcd}
$$

- Tangency encoded in homomorphism of isotropy:

$$\left.\begin{array}{l} \mu_r \longrightarrow \mu_r \\ \zeta_r \longmapsto \zeta_r^{\alpha_{ij}} \end{array}\right\}$$

- Deformation open

- orbifold stable maps $\rightsquigarrow$ orbifold GW invariants.
  Orb(X|D)

[Abramovich-Graber-Vistoli, '06]
[Cadman, '03]

2) Comparisons

- Two theories: complementary features.

- Log more enumerative [N-Ranganathan, '19]
- Orbifold more computable; closer to classical GW [Tseng-You, '20]
- History of Comparisons:

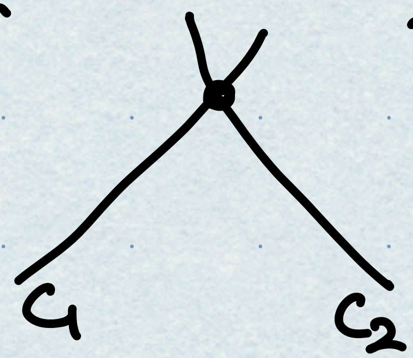
	$D=D_1$ smooth	$D=D_1+\dots+D_k$ snc
$g=0$	Log = Orb on the nose. (universal targets) [Abramovich-Cadman-Wise, '10]	Log $\neq$ Orb for geometric reasons. Log = Orb after passing to a blowup of $(X\mid D)$. (Birational invariance) [BNR, '22]
$g>0$	Orb = Polynomial in r. Log = Constant term [Tseng-You, '18]	???

- Goal: Import orbifold structures to log side (quantum cohomology, torus localisation).

- Missing piece: Comparison compatible with splitting.

3) Splitting and negative tangency

- Boundary of $\mathrm{Orb}(X)$ has recursive structure:

$\{$ 

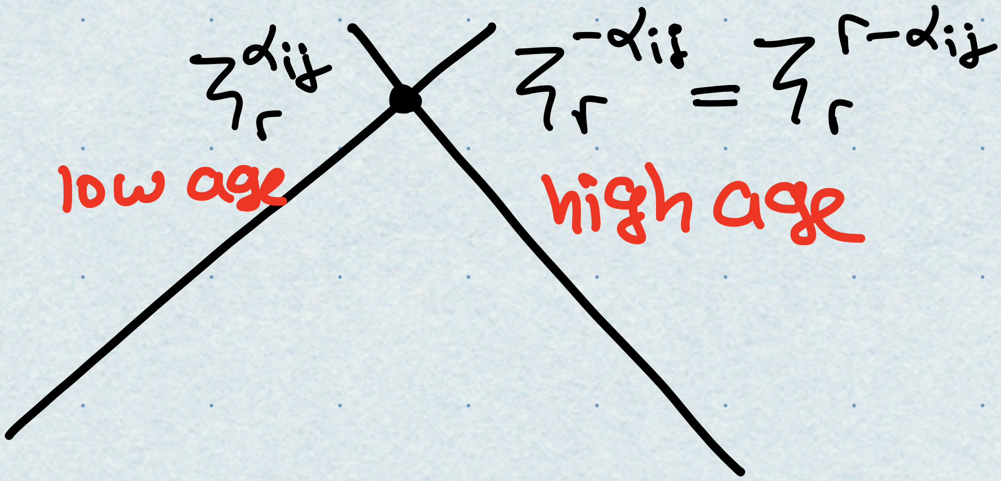
$\}$ $\approx \mathrm{Orb}(X) \times_X \mathrm{Orb}(X)$.

Crucial for all calculations.

- Up to now we have restricted to a special class of orbifold maps:

$$\left.\begin{aligned} \mu_r &\longrightarrow \mu_r \\ \zeta_r &\longmapsto \zeta_r^{\alpha_{ij}} \end{aligned}\right\} \rightsquigarrow \boxed{\text{age} := \frac{\alpha_{ij}}{r} \simeq 0 \quad \text{"low age"}}$$

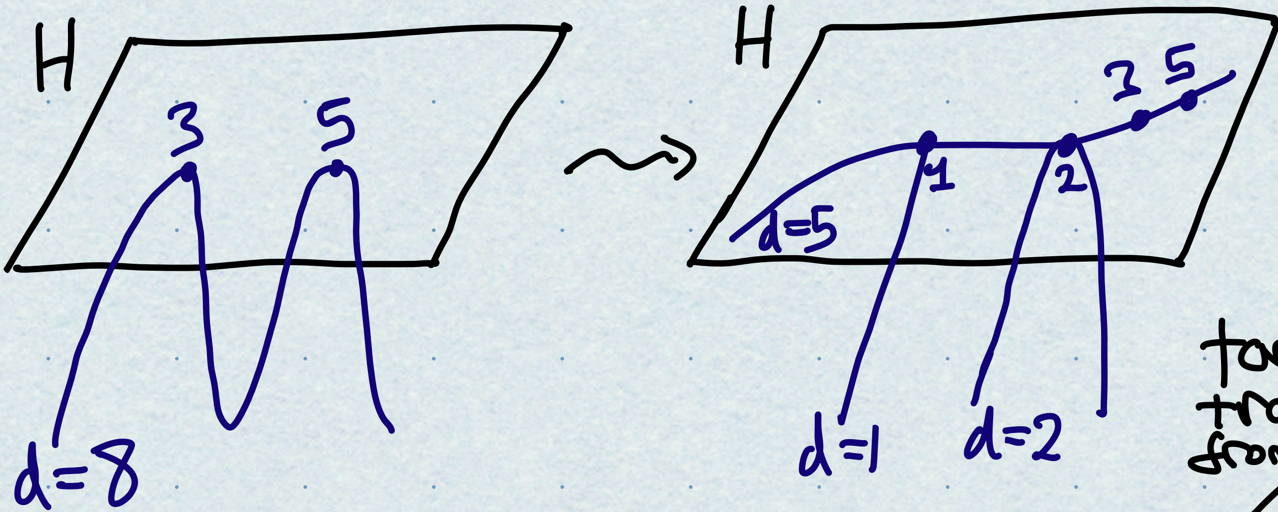
- Across a node, map on isotropy gets inverted:



High age maps enter in splitting formalism.

Need something on log side corresponding to high age.

- Investigate Splitting on log side.
- $(X|D) = (\mathbb{P}^m|H)$, $d=8$, $\alpha=(3,5)$.

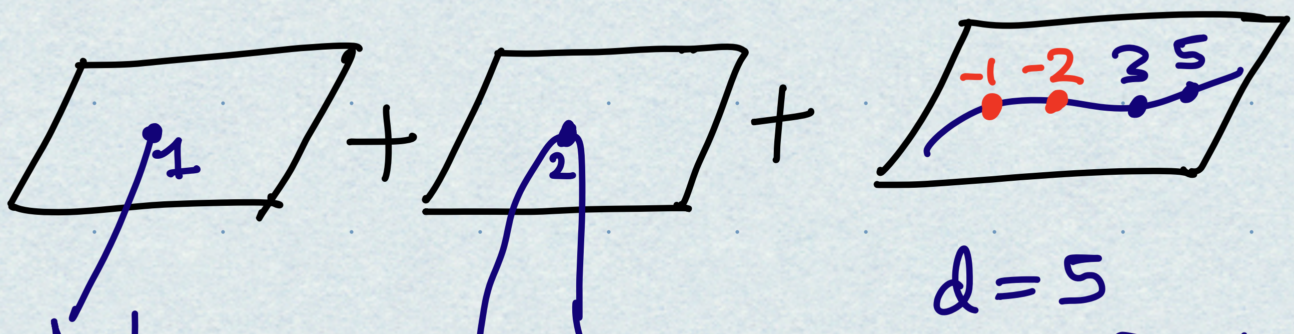


tangency transmitted from nodes

Iff for log lift: $\underbrace{3+5}_{\text{desired tangency at markings}} = \underset{\substack{\uparrow \\ \text{internal degree}}}{5} + 1+2$

(Balancing condition)

Split $\rightsquigarrow$



$d=1$, $\alpha=(1)$ + $d=2$, $\alpha=(2)$ + $d=5$

$3+5 = 5+1+2$

$\Leftrightarrow 3+5-1-2 = 5$

$\boxed{\alpha = (3,5,-1,-2)}$

- Interpret as negative tangency.

(Tangency inverted, same as isotropy.)

"Punctured log GW theory".
[Abramovich-Chen-Gross-Siebert, '20]

- Extend theories:

orbifold w/ low ages $\rightsquigarrow$ orbifold w/ extremal ages (low or high)

log w/ tangencies $\alpha_{ij} \in \mathbb{N}$ $\rightsquigarrow$ log w/ tangencies $\alpha_{ij} \in \mathbb{Z}$

- Upshot will be: we extend our $g=0$, $D=D_1+\ldots+D_k$ comparison to negative tangencies.

  Doesn't look so dramatic.

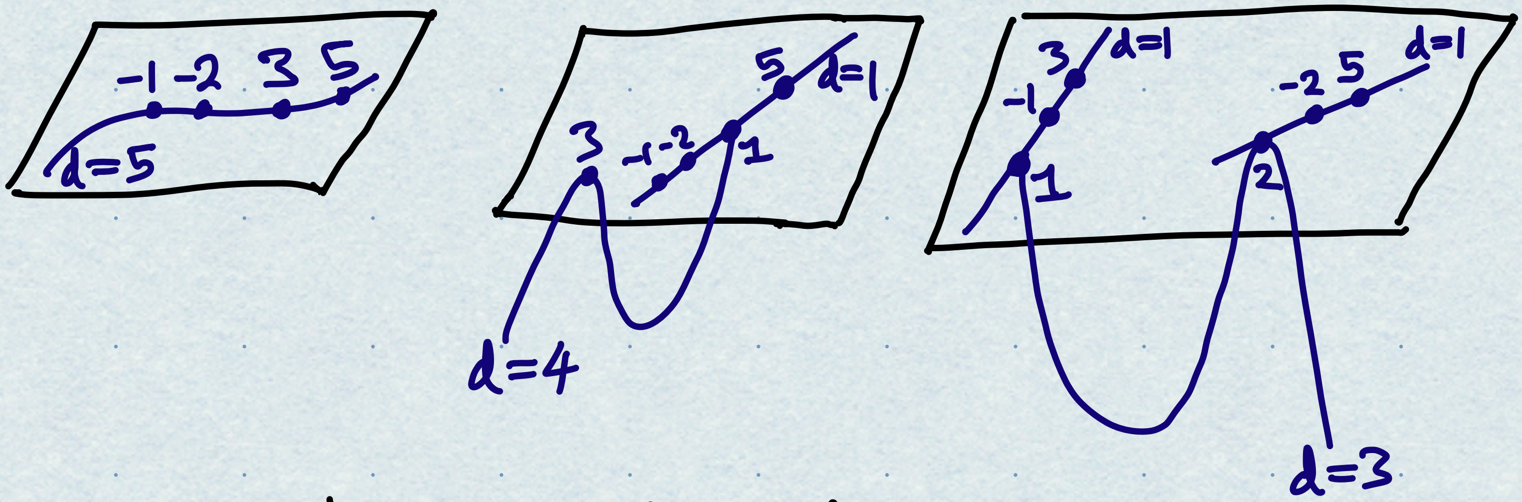
- However, negative tangency moduli space is fundamentally different from positive tangency moduli space.

  Even stating comparison is tricky.

- Return to earlier example:

  $(X|D)=(\mathbb{P}^n|H)$, $d=5$, $\alpha=(3,5,-1,-2)$.

- Rule: each puncture must belong to a curve component mapped inside D.



- Multiple irreducible components
- Happens even virtually!

$$\mathrm{Punct}(X|D) \xrightarrow[\substack{\text{virtually}\\ \text{smooth}}]{\text{POT}} \underbrace{\mathcal{V}(T)}_{\substack{\text{Multiple irreducible cpts}\\ \text{Indexed by tropical types.}}}$$

- $\mathcal{V}(T)$ not even pure-dimensional.
- $\mathcal{V}(T)$ embeds in a smooth stack, cut out by condition that punctures belong to internal components.

Sometimes conditions independent, Sometimes not (not regular embedding).

4) Refined Punctured theory

- ACGS say: Choose an irreducible component of $\nu(T)$, pull back to get a VFC on Punct(X|D).

- We say: there's something else.

  $\nu(T)$ appears as a (non-transverse) intersection of divisors in a smooth space (one for each puncture).

$\Rightarrow$ carries refined intersection class:

$$[\nu(T)]^{ref}.$$

- "Refined virtual class."
  - Pure-dimensional
  - Supported on all irreducible cpts simultaneously.

- Pulls back to $\mathrm{Log}(X|D) \rightsquigarrow$ invariants.
- Intrinsic to punctured theory.

- Th^m A [BNR, '24]: $g=0$, $D=D_1$ smooth:

  Refined Punctured = orbifold on the nose.

- Th^m B [BNR, '24]: $g=0$, $D=D_1+\dots+D_k$ snc:

  Refined Punctured = orbifold after passing to blowup of $(X|D)$.

- Wish I could tell you about proof.

  Instead, some consequences:

- Corollary 1 [BNR '22] : $\mathrm{LogGW}(X|D)$ reconstructible from classical GW of strata.

- Corollary 2 [BNR '24] : Absolute POT for LogGW $\Rightarrow$ torus localisation

- Corollary 3 [Johnston, '24] : New (slightly more general) proof of associativity of Gross-Siebert intrinsic mirror ring (using orbifold WDVV).

Thank you for your attention.