16/09/25

EulerDR talk outline

1. orbifold Euler char. E.g. $\mathbb{P}(1,r)$.
2. Harer-Zagier.
   proof for $g=0$, $g=1$.
3. DR loci.
   Motivation.
   Geometric properties, expected dimension
4. Formula: rank one.
   Formula: higher rank.
5. Proof: rank one.

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Euler chars. of DR loci in genus one

Joint w/ Luca Battistella. (arXiv: 2502.xxx).

① Orbifold Euler characteristic.

- Usual Euler char. satisfies:

(i) cut-and-paste: $Y \subseteq X$:

$$\chi(X) = \chi(X \setminus Y) + \chi(Y).$$

(ii) multiplicativity: $Y \to X$ smooth, fibre F:

$$\chi(Y) = \chi(X) \cdot \chi(F).$$

- But (ii) fails for orbifolds.

$H^*(\mathfrak{X})_{\mathbb{Q}} = H^*(|\mathfrak{X}|)_{\mathbb{Q}}$ coarse space

- Consider: $Pt \to BG = [Pt/G]$.

Universal G-torsor: fibre $\cong G$

But $H^*(BG) = H^*(|BG|) = H^*(Pt)$.

$$\left.\begin{aligned} \Rightarrow \chi(Pt) &= 1 \\ \chi(BG) &= 1 \\ \chi(F) &= |G| \end{aligned}\right\}$$

- Replacement: orbifold Euler characteristic (Sometimes: Stringy)

$\chi_{orb} = \chi$ for Varieties.

Defined so as to satisfy cut-and-paste and multiplicativity for orbifolds.

- E.g. $\chi_{orb}(BG) = \chi_{orb}(Pt) / \chi_{orb}(G)$

$$= 1/|G|.$$

E.g: $\mathbb{P}(1,r)$: coarse space is $\mathbb{P}^1$.

$$\Rightarrow \chi_{orb}(\mathbb{P}(1,r)) = \chi_{orb}(\mathbb{A}^1) + \chi_{orb}(B\mu_r)$$

$$= 1 + \frac{1}{r}.$$

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(2) Harer-Zagier formula

- Consider $M_{g,n}$ as an orbifold.
- Thm (HZ, 1986): we have:

$$\chi_{orb}(M_{g,n}) = \begin{cases} (-1)^{n-3}(n-3)! & g=0,\ n\geq 3 \\ (-1)^n \dfrac{(n-1)!}{12} & g=1,\ n\geq 1 \\ (-1)^n (2g-3+n)! \dfrac{B_{2g}}{2g(2g-2)!} & g=2,\ n\geq 0 \end{cases}$$

- Many proofs. Original: mapping class groups and polygons on surfaces.

  Giacchetto-Lewanski-Norbury 2021: integral of $e(T^{\log})$ on $\overline{M}_{g,n}$ (log Poincaré-Hopf).

- Proof for $g=0$: Here $M_{0,n}$ is a variety, so $\chi_{orb} = \chi$. Induct on n.

  $M_{0,3} = pt \Rightarrow \chi_{orb}(M_{0,3}) = 1$. Works.

  Assume true for $n-1$, consider $M_{0,n} \to M_{0,n-1}$. Fibre is an $(n-1)$-punctured $\mathbb{P}^1$, so:

  $$\begin{aligned} \chi_{orb}(M_{0,n}) &= (2-(n-1)) \cdot \chi_{orb}(M_{0,n-1}) \\ &= (3-n) \cdot (-1)^{n-4}(n-4)! \\ &= (-1)^{n-3}(n-3)! \qquad \square. \end{aligned}$$

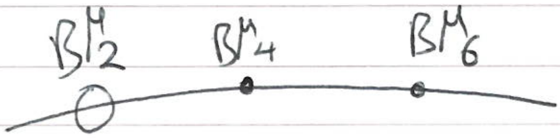
- Proof for $g=1$: Here $M_{1,n}$ is genuinely an orbifold. Again induct on n.

  $M_{1,1}$ = space of elliptic curves.

  Classified by j-invariant $\Rightarrow |M_{1,1}| = \mathbb{A}^1$.

  Generic elliptic curve has order 2 aut.
  $\exists$: one curve w/ order 4 aut
  one curve w/ order 6 aut.

$M_{1,1}$ =



$$\Rightarrow \chi_{orb}(M_{1,1}) = \frac{1}{2}\chi(\mathbb{A}^1 \setminus 2 \text{ pts}) + \frac{1}{4} + \frac{1}{6}$$
$$= -\frac{1}{2} + \frac{1}{4} + \frac{1}{6}$$
$$= -\frac{1}{12}$$

<u>Base case</u> complete

For induction, $M_{1,n} \to M_{1,n-1}$ with fibre a genus one curve minus $(n-1)$ points. Then:

$$\chi_{orb}(M_{1,n}) = -(n-1)\chi_{orb}(M_{1,n-1})$$
$$= -(n-1)\frac{(-1)^{n-1}(n-2)!}{12}$$
$$= \frac{(-1)^n (n-1)!}{12}.$$

$\square$

- Key is formula for M_g.

Harder for $g \geq 2$.

③ DR loci

- Fix $a=(a_1,\dots,a_n)\in\mathbb{Z}^n$ with $\sum_{i=1}^n a_i = 0$

  $DR_{g,n}(a) = \{(C,p_1,\dots,p_n)\in M_{g,n} : \mathcal{O}_C(\sum_{i=1}^n a_i p_i)\cong \mathcal{O}_C\}$
  $\subseteq M_{g,n}$.

- Equivalent to existence of map:

  $f: C\to\mathbb{P}^1$, $f^{-1}(0)=\sum_{a_i>0} a_i p_i$, $f^{-1}(\infty)=\sum_{a_i<0} |a_i| p_i$

- smooth codimension: g. (Except $a=\underline{0}$.)

  unknown if connec

- DR locus, and more precisely its compactification to $\overline{M}_{g,n}$, central to several areas:
  - Hurwitz theory / enumerative geometry.
  - tautological ring of $\overline{M}_{g,n}$: Pixton's formula

- $g=0 \implies DR_{0,n}(a) = M_{0,n}$.

  First nontrivial genus: $g=1$.

4 Results

- ✱ **Thm A [BN]**: $\chi_{orb}(DR_{1,n}(a)) = \frac{(-1)^{n-1}(n-1)!}{24}\left(\sum_{i=1}^{n} a_i^2 - 2\right.$

- Next result concerns higher rank DR loci.
  Given:
  $$A = \begin{pmatrix} a_1^{(1)} & \cdots & a_n^{(1)} \\ & \vdots & \\ a_1^{(r)} & \cdots & a_n^{(r)} \end{pmatrix}$$
  with each row summing to zero, define:
  $$DR^r_{1,n}(A) := \bigcap_{j=1}^{r} DR_{1,n}(a^{(j)}) \subseteq M_{1,n}.$$
  Think of as maps to products of $\mathbb{P}$'s, or more generally to a toric variety of dimension r.

- ✱ **Thm B [BN]**: $\chi_{orb}(DR^r_{1,n}(A)) =$
  $$\frac{(-1)^n}{12} \sum_{k=0}^{r} \sum_{\substack{I \vdash [n] \\ \ell(I)=k+1}} (-1)^k (|I_1|-1)! \cdots (|I_{k+1}|-1)!\; G_{k\times k}(A_I)$$

gcd of $k\times k$ minors.

contraction matrix: sum cols. assoc. to parts of I get $r\times(k+1)$ matrix

[write example families here: see pp. 9-10]

(5) **Proof.** (rank one case) Hinges on a recursi

- Start with $n=2$. Consider forgetful ma

$$DR_{1,2}(a_1,a_2) \xrightarrow{\varphi} M_{1,1}.$$

Given $(C,P_1) \in M_{1,1}$, lift given by choosing $P_2 \in C \setminus P_1$ such that:

$$\mathcal{O}_C(a_1P_1 + a_2P_2) \cong \mathcal{O}_C$$

$$\iff \mathcal{O}_C(a_2P_2) \cong \mathcal{O}_C(-a_1P_1) = \mathcal{O}_C(a_2P_1).$$

There are a_2^2 choices, including $P_2 = P_1$. Removing this, we see:

(*) φ étale of degree $a_2^2 - 1$

$$\Rightarrow \chi_{orb}(DR_{1,2}(a_1,a_2)) = (a_2^2-1)\,\chi_{orb}(M_{1,1}).$$

- In general, consider:

$a = (a_1, \ldots, a_n, a_{n+1})$ with $a_{n+1} \neq 0$.

Consider forgetful map:

$$DR_{1,n+1}(a) \longrightarrow M_{1,n}.$$

$$\chi_{orb} DR^2_{1,4} \begin{pmatrix} a & -a & 0 & 0 \\ 0 & 0 & b & -b \end{pmatrix} = \frac{1}{12}\left(6 - 4a^2 - 4b^2 - 2\gcd(a,b)^2 + 4(ab)^2\right).$$