Roots and Logs 09/05/22

Joint w/ : Battistella-N.-Ranganathan 2203.xxxx

Also : N.-Ranganathan 1908.xxxx

Battistella-N.-Tseng-You 2103.xxxx

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1) Two theories

- $(X \mid D = D_1 + \ldots + D_k)$ snc
- Stable maps w/ tangency c_{ij} to D_i at p_j
- $\mathrm{Nice}(X|D) = \{ f: \mathbb{P}^1 \to X \mid f^* s_{D_i} = \prod_{j=1} s_{p_j}^{c_{ij}} \ \forall i \}$
- Compactification: extra structure on source & target
  1) Log stable maps
     ↳ Log structure: data of monomial functions.
  2) Orbifold stable maps
     ↳ Root stack: gerby structure along D_i and p_j.

     Tangency encoded in homomorphism on isotropy groups.

- Both finite over image in $\overline{M}(X)$.
- For $(\mathbb{P}^r|H)$, both contain $\mathrm{Nice}(\mathbb{P}^r|H)$ as dense open
- Very different flavours, techniques.

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2) Results

- Thm [Abramovich-Cadman-Wise]: $g=0$, $D=D_1$ smooth
  $\Rightarrow$ log theory = orbifold theory
- Two directions to generalise

Source complexity
$g>0$
[redacted]
[Janda-Pandharipande-Pixton-Zvonkine]
[Tseng-You]

Target complexity
$D=D_1+\dots+D_k$ $(k>1)$
us

- Thm [N.-Ranganathan] log $\neq$ orbifold.

Counterexample: $(\mathbb{P}^2|L_1+L_2)$, $d=2$.

Log more enumerative.

- Now for a positive result.
- Th^{m} [Abramovich-Wise]: Log invariants unchanged under strata blowups. (Birational invariance)

  Reasonable: curves to $(X|D)$ are curves in $X \setminus D$ with prescribed asymptotics.

- Th^{m} [Battistella-N.-Ranganathan]: Birational invariance is key property distinguishing log and orbifold theories.

  For each choice of discrete data, $\exists$ iterated blowup

  $$(X^{+}|D^{+}) \longrightarrow (X|D)$$

  such that $\mathrm{Orb}(X^{+}|D^{+}) = \mathrm{Log}(X^{+}|D^{+}) \left(\overset{[AW]}{=} \mathrm{Log}(X|D) \right)$

  Orbifold theory then stable under further blowups. ("Limit orbifold theory")

- Blowing up target, instead of moduli space, cf. N.-Ranganathan and works on DR cycle (Holmes-Pixton-Schmitt, Molcho-Ranganathan, ...).
- Imports orbifold techniques.

  $\hookrightarrow$ WDVV, localisation, boundary recursion, ...

- Corollary [BNR]: Log theory of $(X|D)$ reconstructable from absolute theories of Strata. (Topological view.)

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3) Rank reduction

- Idea: bootstrap result for $(X|D = D_1 + \dots + D_k)$ from result for smooth pairs $(X|D_i)$.
- Set $k=2$ to ease notation.

  Almost all phenomena visible here.
- Thm [BNTY, BNR]: orbifold theory satisfies a product rule over $\overline{M}(X)$:

$$\begin{tikzcd} \mathrm{orb}(X|D_1+D_2) \arrow[r] & F \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{orb}(X|D_1) \arrow[d] \\ & \mathrm{orb}(X|D_2) \arrow[r] & \overline{M}(X) \end{tikzcd}$$

Respecting VFCs. So:

$$[\mathrm{orb}(X|D_1+D_2)]^{\mathrm{virt}} = [\mathrm{orb}(X|D_1)]^{\mathrm{virt}} \cdot [\mathrm{orb}(X|D_2)]^{\mathrm{virt}} \text{ in } A_*(\overline{M}(X))$$

- Q: Does same hold for log?

  (If yes, would get simple generalisation)
- A: No!

  (Similar story for DRC.)

- Def^n [N., N.-Ranganathan] Naive space is:

$$\begin{tikzcd}
\mathrm{Naive}(X|D) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Log}(X|D_1) \arrow[d] \\
\mathrm{Log}(X|D_2) \arrow[r] & \overline{M}(X)
\end{tikzcd}$$

- Log = Orb for smooth divisors + product rule

$\Rightarrow$ naive = orbifold always.

- Question becomes: when does naive = log?

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4) Counterexample

- Often (universal geometry, $\mathbb{P}^r | H_1 + H_2, \ldots$) Log(X|D) is irreducible, contains nice locus as dense.
- Naive(X|D) almost never irreducible.

Has excess components.

Contains Log(X|D) as "main component."

Hence log "more enumerative" than naive/orbifold

- $$\underbrace{\overline{\mathrm{Nice}(X|D_1) \cap \mathrm{Nice}(X|D_2)}}_{\mathrm{Log}(X|D)} \subsetneq \underbrace{\overline{\mathrm{Nice}(X|D_1)} \cap \overline{\mathrm{Nice}(X|D_2)}}_{\mathrm{Naive}(X|D)}$$

- Gathmann: intersection-theoretic criterion which determines image of $\mathrm{Log}(X|D_i) \to \overline{M}(X)$.

  $\Rightarrow \exists$ simple criterion for image$(\mathrm{Naive}(X|D) \to \overline{M}(X))$.

  $\Rightarrow$ Can work with naive space concretely.

- Example [N., N.-Ranganathan]:

$(X|D) = (\mathbb{P}^2|L_1+L_2)$

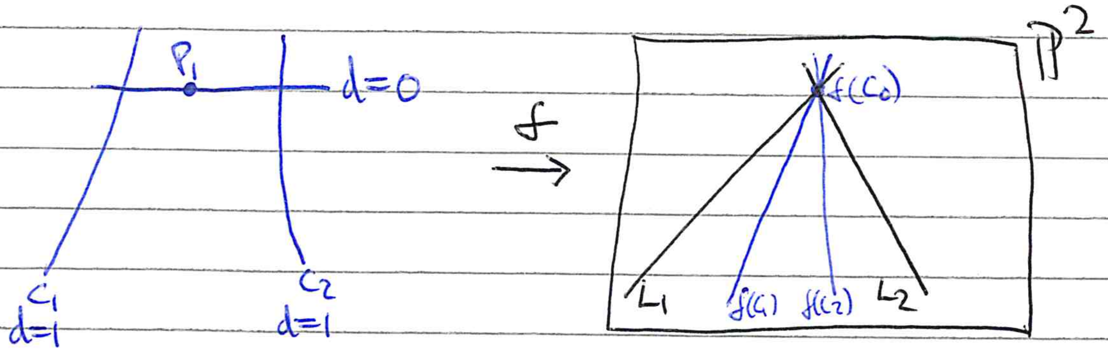
$d = 2$

1 marking P_1, tangency $\binom{2}{2}$

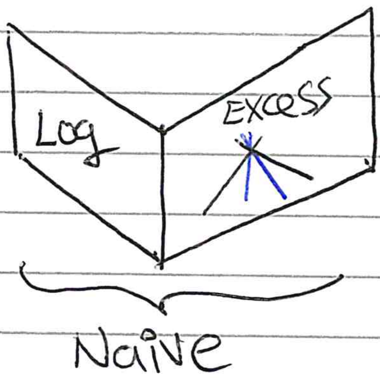
$\Big\}$ vdim $= 5-2-1 = 2$.

- Fact: $(\mathbb{P}^2|L_1+L_2)$ convex $\Rightarrow \mathrm{Log}(\mathbb{P}^2|L_1+L_2)$ is:
  - irreducible
  - closure of nice locus
  - dimension 2.

- Look at stable maps to $\mathbb{P}^2$ of form:

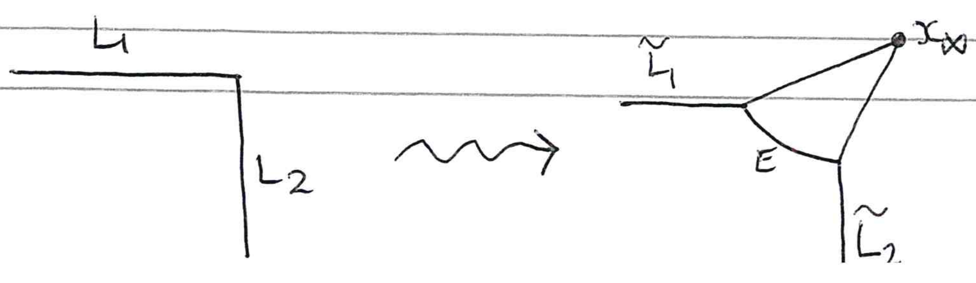


- Easy check: lifts to expanded map to $(\mathbb{P}^2|L_1)$ and $(\mathbb{P}^2|L_2)$ separately. (or: Gathmann condition satisfied)
  $\Rightarrow$ defines locus in Naive$(\mathbb{P}^2|L_1+L_2)$.
  dimension of locus is 2.
- Naive space has 2 irreducible components:

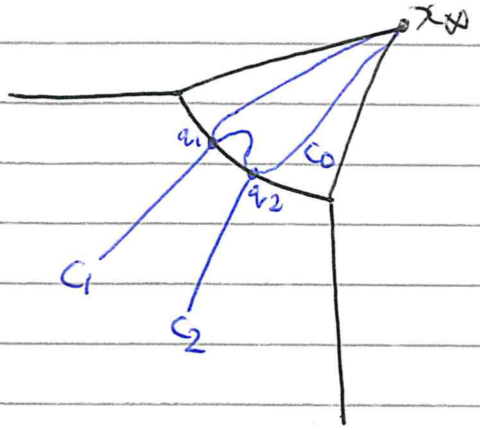


- [Naive] = [Log] + [Excess]
  $\Rightarrow$ invariants differ (cf. N.-Ranganathan for explicit insertions.)

- How to make sense of intersection/failure of log lift?
- Via expansions: degenerate to normal cone:



- Exists lift to $\mathrm{Log}(\mathbb{P}^2|L_1+L_2)$ iff exists predeformable lift to expansion:



- C_0 conic in $\mathbb{P}^2$ tangent to both branches at x_∞

  $\Rightarrow$ C_0 double line

  $\Rightarrow$ $f(q_1) = f(q_2)$ in E

  $\Rightarrow$ original lines $f(C_1), f(C_2)$ needed to be the same line.

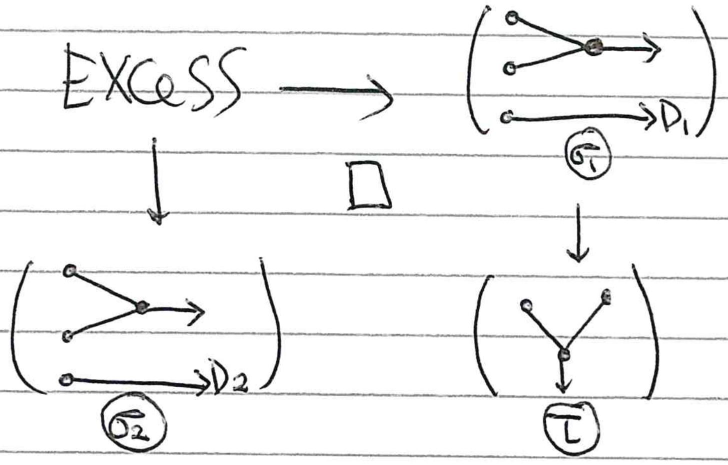
- Via log structures

  The log maps to $(\mathbb{P}^2|L_1), (\mathbb{P}^2|L_2)$ define log enhancements e_1, e_2 of C.

  These have a $\mathbb{G}_m$ worth of moduli each.

  Need enhancements to be the same.

- <u>Aside: identifying excess loci.</u>
- Excess locus in naive space is fibre product of boundary strata, indexed by cones:



- Can read off excess dim from cone dims:

$$\text{exc. dim.} = \dim\tau - \dim\sigma_1 - \dim\sigma_2 .$$

- Along strata, intersection $\mathrm{Log}(X|D_1)\cap \mathrm{Log}(X|D_2)$ non-transverse:

$$\underbrace{\overline{\mathrm{Nice}(X|D_1)\cap \mathrm{Nice}(X|D_2)}}_{\mathrm{Log}(X|D)} \subsetneq \underbrace{\overline{\mathrm{Nice}(X|D_1)}\cap \overline{\mathrm{Nice}(X|D_2)}}_{\mathrm{Naive}(X|D)}$$

Also can be understood via cones.

- Different approach: blowup $\overline{M}(X)$ to transversalise intersection.

- Tropical dictionary allows us to organise as strata blowup.

  Cf. N.-Ranganathan, where given as explicit iterated blowup.

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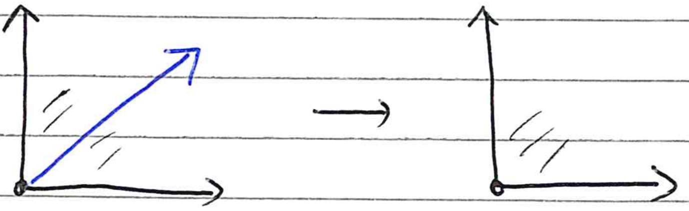
5) Target blowups

- Return to counterexample.
- Blowing up $L_1 \cap L_2$ in $\mathbb{P}^2$ forces lines $f(C_1), f(C_2)$ to be same, even in naive space.

  $\Longrightarrow$ excess locus dissappears

  $$\text{Naive} = \log.$$

- Tropically blowup is:



- Th^m [BNR]: Subdivide $\Sigma(X|D)$ by introducing rays for all tangency vectors (slopes) appearing in all tropical curves. Get

$$(X^+|D^+) \longrightarrow (X|D).$$

Then $\text{Naive}(X^+|D^+) \simeq \text{Log}(X^+|D^+)$.

- Pf: Construct an expansion, and a predeformable lift to it.

  Note $X^+ \to X$ really a blowup, not an expansion.

  Passing to blowup does not give complete transversality of $f: C \to X^+$. E.g. map can still intersect strata of codim ≥ 2.

  But constrains behaviour enough to obtain result. $\square$

- Upcoming Projects:

  (i) Version for negative contacts.

  Requires one new idea, inspired by work of Fan-Wu-You.

  (ii) Higher genus.

  Need universal geometry version of Tseng-You / JPPZ comparison for smooth divisors.