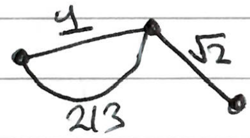
Universality talk 03/02/2022 (Frankfurt)

Joint w/ Gabriel Corrigan
Dan Simms

1) Tropical maps

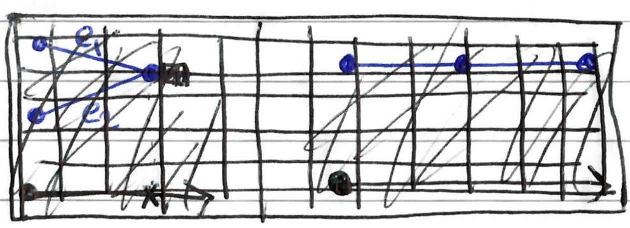
Warning: my rules might differ from yours. Mine are adapted to tropicalising stable logarithmic maps to arbitrary pairs. But differences are not really important.

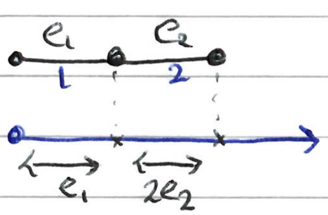
Tropical curve: Metrised graph:

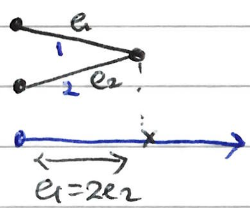


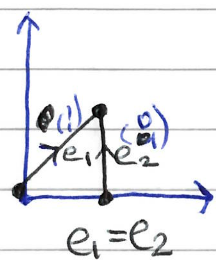
Target: $\mathbb{R}^n_+ = \mathbb{R}^n_{\geq 0}$ considered as RPC.

Maps $f: \Gamma \to \mathbb{R}^n_+$ with integral slopes/metric expansion factors.









Experts: no balancing condition. Will explain why later. For our results, not a big deal.

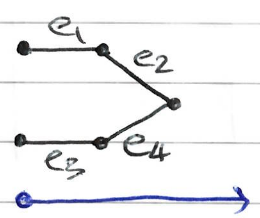
Tropical type: forget edge lengths.
Remaining data is combinatorial:

- Source graph Γ.
- Slope vectors $m_{\vec{e}} \in \mathbb{Z}^n$
- $v \in V(\Gamma) \rightsquigarrow \sigma_v \leq \mathbb{R}^n_+$
  $e \in E(\Gamma) \rightsquigarrow \sigma_e \leq \mathbb{R}^n_+$.

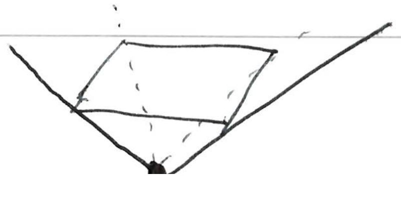
Associated to this data, get corresponding tropical moduli space: parametrising metric enhancements of Γ compatible with the type. In above examples:

$(\mathbb{R}^2_+)_{e_1 e_2}$ $(\mathbb{R}_+)_{e_2}$ $(\mathbb{R}^+)_{e_1}$

Always an RPC. But not always smooth:



$(e_1 + e_2 = e_3 + e_4) \subseteq (\mathbb{R}^4_+)_{e_1 e_2 e_3 e_4}$



Toric variety: $V(x_1 x_2 - x_3 x_4) \subseteq \mathbb{A}^4_x$

2) Logarithmic maps

X smooth projective

$$M(X) = \left\{ \begin{matrix} C \xrightarrow{f} X \\ p_1, \dots, p_m \in C^{sm} \end{matrix} \;\middle|\; \begin{matrix} f \text{ has finitely} \\ \text{many auts} \end{matrix} \right\}$$

a.w. nodal

Thm (Vakil): This class of moduli spaces exhibits arbitrary singularities.

This is bad. But, all $M(X)$ are virtually smooth: $\exists$ POT which controls deformation theory

$\Rightarrow$ VFC and GW invariants.

$\begin{cases} \text{torus localisation} \\ \text{HRR} \\ \text{etc.} \end{cases}$

Log stable maps: inspired in part by tropical correspondence theorems.

$(X \mid D = D_1 + \dots + D_n) \longleftarrow$ toroidal embedding.

smooth, snc

$\Sigma_{X|D} = \mathbb{R}^n_+$

Power of fan (subdivisions, PL fns).

Study curves in X w/ fixed tangency profile along D_i. Encode in matrix α.

$$\mathrm{Log}_\alpha(X|D) = \left\{ \begin{matrix} C \xrightarrow{f} X \\ p_1, \ldots, p_m \in C^{sm} \end{matrix} \;\middle|\; f^* S_{D_i} = \prod_{j=1}^{m} S_{p_j}^{\alpha_j^i} \right\}$$

As with $M(X)$, $\mathrm{Log}_\alpha(X|D)$ arbitrarily singular.

Unlike $M(X)$, $\mathrm{Log}_\alpha(X|D)$ only virtually irreducible.

Has POT over $\mathrm{Log}_\alpha(\mathbb{A}^n|\partial\mathbb{A}^n)$.

↳ Irreducible but not smooth.

$\Rightarrow$ VFC and GW invariants OK.

$\left.\begin{matrix} \text{Torus localisation} \\ \text{HRR} \\ \text{etc.} \end{matrix}\right\}$ not ok.

<u>Goal</u>: understand singularities of $\mathrm{Log}_\alpha(\mathbb{A}^n|\partial\mathbb{A}^n)$

("virtual singularities" of $\mathrm{Log}_\alpha(X|D)$.)

<u>Fact</u> (log deformation theory): $\mathrm{Log}_\alpha(\mathbb{A}^n|\partial\mathbb{A}^n)$ a toroidal embedding.

$\Rightarrow$ singularities at worst toric.

virtual toric universality.

Thm A (Corrigan-N-Simms): For each toric singularity, $\exists n \in \mathbb{N}$ s.t. the singularity appears on $\mathrm{Log}_\alpha(\mathbb{A}^n|\partial\mathbb{A}^n)$.

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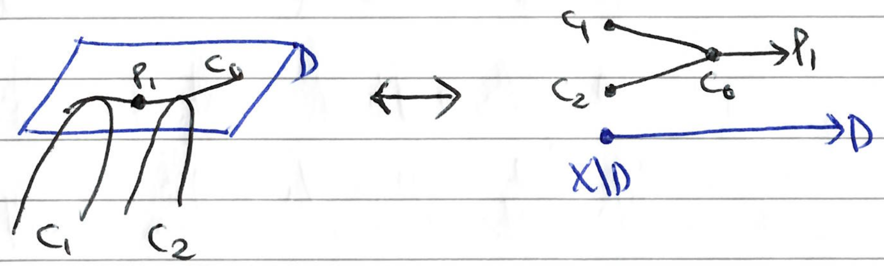
3) **Faithful tropicalisation** (combining 1 and 2)

$\mathrm{Log}_\alpha(\mathbb{A}^n|\partial\mathbb{A}^n)$ toroidal embedding.

Boundary is where curve breaks.

E.g.:



$$\Rightarrow \left\{\begin{array}{c}\text{boundary strata}\\ \text{in } \mathrm{Log}_\alpha(\mathbb{A}^n|\partial\mathbb{A}^n)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{tropical types}\\ \text{of maps to } \mathbb{R}^n_+\end{array}\right\}$$

$$\left(\begin{array}{c}\text{local toric model}\\ \text{of } \mathrm{Log}_\alpha(\mathbb{A}^n|\partial\mathbb{A}^n) \text{ at}\\ \text{generic pt of boundary}\\ \text{stratum}\end{array}\right) = \left(\begin{array}{c}\text{affine}\\ \text{toric variety}\\ \text{associated to}\\ \text{tropical moduli}\\ \text{space}\end{array}\right)$$

Here, Thm A reduces to:

Thm B (CNS): For every RPC σ, $\exists n \in \mathbb{N}$ and a tropical type of map to $\mathbb{R}^n_+$, whose corresponding tropical moduli space is isomorphic to σ.

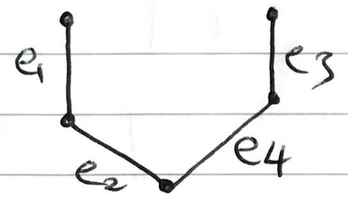
4) Combinatorial Proof

Given $\sigma \subseteq N_{\mathbb{R}}$, consider monoid:

$P = \sigma^{\vee} \cap M$

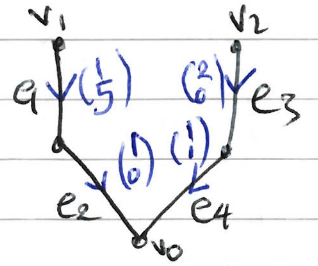
Will phrase in terms of this.

E.g.: $P = \mathbb{N}^4 / (e_1 + e_2 = 2e_3 + e_4,\ 5e_1 = e_4)$.

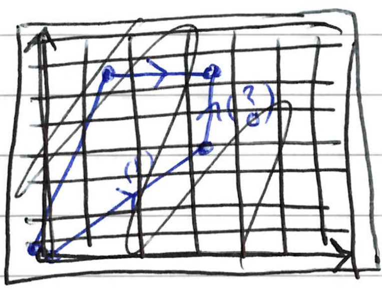


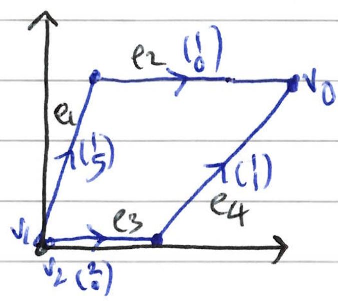
$n = 2 = \#$ relations.

Slope along e = vector of coeffs. of e in rel



Picture is:





x-dir: $e_1 + e_2 = 2e_3 + e_4$

y-dir: $5e_1 = e_4$

General case:

① Take monoid P with presentation:

$$P = \mathbb{N}^G / R.$$

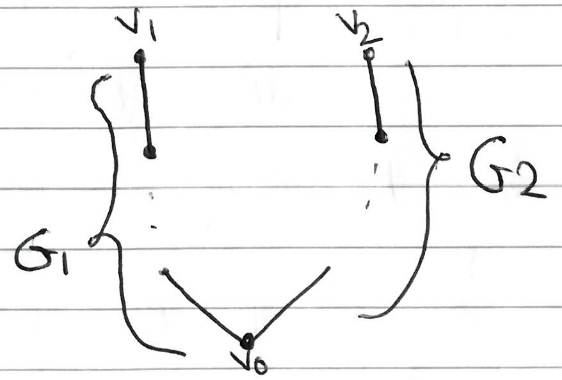
② Modify to bipartite presentation:

$$G = G_1 \sqcup G_2$$

G_1: in LHS of rels only

G_2: only in RHS of rels

③



④ $n = |R|$

Relations encoded in slope vectors.

Note for experts: lots of subtleties involving integralisation, saturation etc. Don't worry: we take care with these.

5) Boundedness

Above construction: n depends on P.
(It's $=$ # rels. in bipartite presentation)

Question: Does $\exists$ uniform n?

Th^m C [CNS]: $n=1$ does not work.

Cone over K-gon ($K \geq 7$) does not appear in ~~[illegible]~~ $\mathrm{Trop}(\mathbb{R}_+)$.