Prym Tautological Talk

Outline:

1) A_g and $\overline{A}_g$
2) Tautological ring
3) Torelli cycle
4) Prym ~~[illegible]~~ cycle
5) GRR
6) Algorithm on $\overline{H}_g$.

(Joint w/ Yoav Len and Sam Molcho.)

1) A_g and $\overline{A}_g$

- $A = V/\Lambda$ abelian variety of $\dim = g$.

  [E.g. $A = \mathrm{Jac}(C)$.]

- Principal polarisation: homological equivalence class of well-behaved ample divisor.

  (Rigidifies. Also $\exists$ non-algebraic $\mathbb{C}$-tori.)

  [E.g. Θ on $\mathrm{Jac}(C)$. Needed to recover C.]

- $\boxed{A_g}$ = moduli space of PPAV of dim g.

- Construction: $A = \mathbb{C}^g/\Lambda$ with:

  $\Lambda = \mathbb{Z}^g + \boxed{Z}\,\mathbb{Z}^g$ $\quad$ $\mathbb{Z}$-column span of Z

  with $Z \in \mathrm{Mat}_{g\times g}(\mathbb{C})$:

  - Z symmetric
  - $\mathrm{Im}\,Z$ positive definite.

  $\mathbb{H}_g$ = space of Z = "Siegel upper half space"

  $A_g = \mathbb{H}_g / \mathrm{Sp}(2g, \mathbb{Z})$

- $\dim A_g = \dim \mathbb{H}_g = \boxed{\binom{g+1}{2}}$.

  A_g smooth DM stack, non-proper.

[Alternative construction via Hilbert schemes:
$$\varphi_{L^3} : A \hookrightarrow \mathbb{P}^{3^g - 1}$$
]

- ~~Compactification~~: there are many.
- Hiding details/technicalities, there is a moduli space

A_g^{trop}

[Ash-Mumford-Rapoport-Tai '75]
[Faltings-Chai '90]

with no preferred cone complex structure:

$$\text{Choice } (\Sigma \to A_g^{trop}) \rightsquigarrow \left(\boxed{\overline{A}_g^{\Sigma}} \supseteq A_g \right) \text{ toroidal compactification}$$

- What does $\overline{A}_g^{\Sigma}$ parametrise? Fiddly.

Semiabelian varieties (plus extra data): "Degeneration data"

$$0 \to T \to G \to X \to 0 \qquad (*1)$$

$T \cong \mathbb{G}_m^r$; X abelian, $\dim = g - r$.

[Néron '61, Grothendieck '72]

[E.g.: If $C \rightsquigarrow C_0$ by acquiring r nodes, then $\mathrm{Jac}(C) \rightsquigarrow \mathrm{Jac}(C_0)$:
$$0 \to \mathbb{G}_m^r \to \mathrm{Jac}(C_0) \to \mathrm{Jac}(\tilde{C}_0) \to 0.]$$

- Every $\overline{A}_g^{\Sigma}$ carries a universal semiabelian variety:

(*2) $$\boxed{\mathcal{U}_g^{\Sigma} \underset{e}{\overset{q}{\rightleftarrows}} \overline{\mathcal{A}}_g^{\Sigma}}$$ (Stable under base change to a refinement.)

- Some carry a compactified universal family.

  Idea: Compactify (*1) to ~~[illegible]~~ on expansion.

  Mumford degeneration: Polyhedral decomposition of real torus.

  Can reverse logic and use this to construct Σ, by combinatorial SS reduction.

- [Strata of $\overline{\mathcal{A}}_g^{\Sigma}$ explicit: torus bundle over abelian variety bundle over $\mathcal{A}_{g-r}$.]

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2) Tautological ring

- Look back at (*2). Define:

$$\boxed{\begin{aligned} \mathbb{E} &= e^*\Omega_q \\ \lambda_i &= c_i(\mathbb{E}) \end{aligned}}$$ (Hodge bundle, $\mathrm{rk}=g$.)

[Torelli: $M_g \xrightarrow{Tor} A_g$, $\mathrm{Tor}^* \mathbb{E} = \mathbb{E}$]
$C \mapsto \mathrm{Jac}(C)$

- Hodge bundle preserved under pullback along $\overline{A}_g^{\Sigma'} \to \overline{A}_g^{\Sigma}$

- <u>Def^n</u>: The tautological ring

$$\boxed{R^*(\overline{A}_g)} \subseteq CH^*(\overline{A}_g).$$

is the subring generated by $\lambda_1, \dots, \lambda_g$.
Doesn't depend on Σ (dropped from notation).

Relatively simple, e.g. no boundary classes.
In fact:

[Van der Geer '96]

- <u>Th^m</u> [vdG]: As a graded ring:

$$\boxed{R^*(\overline{A}_g) = \mathbb{Q}[\lambda_1, \dots, \lambda_g] / ((1+\lambda_1 + \dots + \lambda_g)(1-\lambda_1 + \dots + (-1)^g \lambda_g))}$$

(Equivalent to $ch_{2k}(\mathbb{E}) = 0$ for $k \geq 1$.)

<u>Pf</u>: GRR for universal Θ divisor to find relation.

Use $\lambda_1 \cdots \lambda_g \neq 0$ and properties of Gorenstein rings to show this is only relation. $\square$

$\Rightarrow$ $R^*(\overline{\mathcal{A}}_g)$ based by squarefree monomials in λ_i.

$\Rightarrow$ $R^*(\overline{\mathcal{A}}_g)$ Gorenstein: restricting of intersection pairing is perfect.

[Not automatic: $R^*(\overline{\mathcal{M}}_{g,n})$ is rarely Gorenstein, cf. Canning]

- Produces tautological projection:

$$\gamma \in CH^k(\overline{\mathcal{A}}_g) \rightsquigarrow \langle \gamma, - \rangle : R^{\binom{g+1}{2}-k}(\overline{\mathcal{A}}_g) \to \mathbb{Q}$$

$$\rightsquigarrow \boxed{\mathrm{taut}(\gamma) \in R^k(\overline{\mathcal{A}}_g).}$$

- Goal: Compute tautological projection of natural cycles on $\overline{\mathcal{A}}_g$.

[Our techniques also apply to the tautological projection of $\mathcal{A}_g$, cf. Canning-Molcho-Oprea-Pandharipande.]

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3) Torelli cycle

- Torelli map extends:

$$\overline{\mathcal{M}}_g \xrightarrow[\mathrm{Tor}]{} \overline{\mathcal{A}}_g^{\Sigma}, \qquad \mathrm{Tor}^* \lambda_i = \lambda_i$$

(Sometimes need to blow up $\overline{\mathcal{M}}_g$; it's a question of whether tropical Torelli respects the cone complex structures. But not necessary for Σ = 1st or 2nd Voronoi, for what it's worth.)

- Computing taut($Taut_*[\overline{M}_g]$)

$$\updownarrow$$

Computing $\int_{\overline{M}_g} \lambda_{i_1} \cdots \lambda_{i_k}$ (squarefree monomial)

- Faber (1997): algorithm (and code) for this.

Uses Mumford's GRR formula for λ_i.
Boundary classes appear $\Rightarrow$ recursive.

(*) Key trick: Forget markings at end of each recursion step.

Valid since λ_i pulled back from $\overline{M}_{g,0}$.

Crucial to avoid explosion of combinatorial complexity. (Remember this!).

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4) Prym cycle

- Jacobians nicest ppavs.
Pryms second-nicest.

- $f: C \to B$ étale double cover of smooth curves.

$f^* Jac(B) \subseteq Jac(C)$ [Mumford '74]

$\boxed{Prym(f)} \subseteq Jac(C)$ complementary abelian subvariety.

- $0 \to \mathrm{Prym}(f) \to \mathrm{Jac}(C) \xrightarrow{\mathrm{Nm}_f} \mathrm{Jac}(B) \to 0$ (up to extension by $\mathbb{Z}/2\mathbb{Z}$)

- $\underbrace{f^*\mathrm{Jac}(B)}_{\substack{+1 \text{ e.space} \\ \text{for } \tau}} \times \underbrace{\mathrm{Prym}(f)}_{\substack{-1 \text{ e.space} \\ \text{for } \tau}} \twoheadrightarrow \mathrm{Jac}(C)$

- $$\begin{aligned} \dim \mathrm{Prym}(f) &= \dim \mathrm{Jac}(C) - \dim \mathrm{Jac}(B) \\ &= g_C - g_B \\ &= g_B - 1 \qquad (RH \Rightarrow g_C = 2g_B - 1) \end{aligned}$$

[Why double covers? why étale?
Don't really need, but then polarisation isn't principal.
Our work applies, but formula less nice.]

- $\boxed{H_g}$ = Hurwitz space of étale double covers $C \to B$, $g_B = g$.

[$t: H_g \to M_g$ finite, degree $= (2^{2g}-1)/2$]

- $$\left.\begin{array}{l} H_g \xrightarrow{\mathrm{Prym}} A_{g-1} \\ (C \xrightarrow{f} B) \mapsto \mathrm{Prym}(f) \end{array}\right\}$$

By tropical geometry, lifts to compactification:

$$\overline{H}_g^{\Sigma} \xrightarrow[\mathrm{Prym}]{} \overline{A}_{g-1}^{\Sigma}$$

[Unlike Torelli, actually have to blowup $\overline{H}_g$ in all known cases.]

- Computing $\mathrm{taut}(\mathrm{Prym}_*[\overline{H}_g])$

$$\updownarrow$$

Computing $\displaystyle\int_{\overline{H}_g} \mathrm{Prym}^*\lambda_{i_1} \cdots \mathrm{Prym}^*\lambda_{i_K}$

These are prym lambda classes; consider

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[dr] & B \arrow[d] \\ & \overline{H}_g^{I} \end{tikzcd}$$

There are three associated abelian varieties, each with their own Hodge bundle:

$\mathrm{Jac}(C)$	$\mathbb{E}_C$
$\mathrm{Jac}(B)$	$\mathbb{E}_B$
$\mathrm{Prym}(f)$	$\widetilde{\mathbb{E}}$

They are related:

$$0 \to \widetilde{\mathbb{E}} \to \mathbb{E}_C \to \mathbb{E}_B \to 0$$

and:

$$\boxed{\mathrm{Prym}^*\lambda_i = c_i(\widetilde{\mathbb{E}})}$$

- All well-defined on $\overline{\mathcal{H}}_g$, so can discard Σ and work there. (Projection formula.)
- <u>Idea</u>: Use projection formula for $t: \overline{\mathcal{H}}_g \to \overline{\mathcal{M}}_g$.

$$\tilde{\mathbb{E}} = \mathbb{E}_C - \mathbb{E}_B = \mathbb{E}_C - t^*\mathbb{E}$$

- <u>Hope</u>: Express $\mathbb{E}_C$ in terms of $\mathbb{E}_B$. Then everything difficult will be pulled back via t.

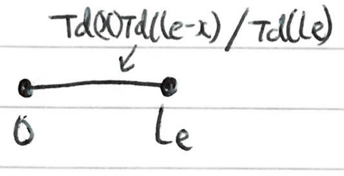
<u>5) GRR</u>

- $$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "p_C"'] & & B \arrow[dl, "p_B"] \\ & \overline{\mathcal{H}}_g & \end{tikzcd}$$

$$\left.\begin{aligned} \mathbb{E}_C &= p_{C*}\omega_{p_C} \\ \mathbb{E}_B &= p_{B*}\omega_{p_B} \end{aligned}\right\}$$ Compute each using GRR and compare

f log etale $\Rightarrow f^*\omega_B = \omega_C$.

Key difference is Todd class: $Td(T_{p_C})$ vs. $Td(T_{p_B})$.



(Piecewise power series).

$\Rightarrow$ $Td(T_{p_C}) \neq Td(T_{p_B})$ only along locus of nodes where f is ramified.

~~Key simpl~~

For etale double covers this locus is simple:

- Ramified $\Rightarrow$ mult. 2
- No ramification $\Rightarrow$ node cannot be separating (RH parity)
  at markings

$\Rightarrow$ only one boundary divisor contributes:

$$\boxed{\Delta_0 = \left[\begin{array}{c} {}_{2g-2}\bigcirc\!\!\!\!\!{}^{\;\;\;\;\;\;2} \\ \downarrow \\ \bigcirc_{g-1} \end{array}\right]}, \qquad \begin{array}{l} L : \overline{H}_{g-1,(2)^2} \to \overline{H}_g \\ L_*(1) = 2\Delta_0 \end{array}$$

Allows us to compare $\mathbb{E}_C$ and $\mathbb{E}_B = t^*\mathbb{E}$.

Th^m: In $\overline{H}_g$ we have:

$$\mathrm{ch}(\tilde{\mathbb{E}}^\vee) = t^*\mathrm{ch}(\mathbb{E}^\vee) - 1 + \frac{1}{2} L_*\left(\sum_{k\geq 1} \frac{(2^{2k}-1)B_{2k}}{(2k)!}\left(\psi_1^{2k-2} - \psi_1^{2k-3}\psi_2 + \dots + \psi_2^{2k-2}\right)\right)$$

For μ_2, equivalent to Chiodo's formula. But ours applies to arbitrary G-covers. Already different from Chiodo for $G = \mu_r$.

Essentially, poly appearing in Mumford's GRR calculation. Universal polynomial for Todd classes of codim. 2 loci

6) Implementation

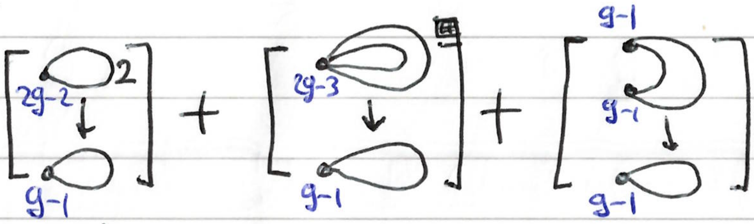
- Remains to integrate chern classes of $\tilde{\mathbb{E}}$.

Same idea: Projection formula for t.

- **Problem 1**: $\Delta_0 \neq t^*\delta_0$.

In fact:

$t^*\delta_0 = 2[\;\;] + [\;\;] + [\;\;]$



(Δ_0)

(Can prove using extended piecewise polynomials)

- So we have to work directly on $\overline{H}_g$.

We can do this, but the combinatorics is complicated. Which brings us to:

- **Problem 2**: There is no forgetful map:

$\overline{H}_{g-1,(2)^2} \longrightarrow \overline{H}_{g-1}$.

Thus, unlike Faber, we can't control the number of markings in the recursion. The combinatorics explodes. (g = 6 is base case!)

~~Still, hope to finagle our way to one working example. Perhaps I can report on this in a future talk.~~

- Solution: Instead of multiplying out RHS directly, use invariance of lambda classes:

$$\begin{cases} L^\vee \mathrm{ch}^\vee(\widetilde{\mathbb{E}}) = \mathrm{ch}^\vee(\widetilde{\mathbb{E}}) & \text{(loop gluing)} \\ B_\Gamma^\vee \mathrm{ch}^\vee(\widetilde{\mathbb{E}}) = \mathrm{ch}^\vee(\widetilde{\mathbb{E}}_1) + \mathrm{ch}^\vee(\widetilde{\mathbb{E}}_2) & \text{(bridge gluing)} \end{cases}$$

This vastly improves the efficiency.

- The algorithm is computer-implemented and effective for $g \leq 10$.

(obstacle for $g \geq 11$ is actually enumerating the basis of $R^*(A_{g-1})$).

- In $g=6$ (first nontrivial case) we recover Donagi's result:

$$\mathrm{Prym}_*[\bar{R}_6] = 27 \cdot [\bar{A}_5]$$

- In $g=7$ we ~~[illegible]~~ find:

$$\boxed{\mathrm{taut}\,\mathrm{Prym}_*[\bar{R}_7] = \frac{691}{178{,}541{,}140{,}377{,}600}(10\lambda_1\lambda_2 + \lambda_3)}$$

- In $g=8$ we find (PTO):

$$\mathrm{taut\,Prym}_{\chi}[F_{18}] =$$
$$- \frac{691}{158{,}703{,}235{,}891{,}200}\lambda_7$$
$$+ \frac{69{,}791}{856{,}997{,}473{,}812{,}480}\lambda_1\lambda_6$$
$$- \frac{131{,}981}{856{,}997{,}473{,}812{,}480}\lambda_2\lambda_5$$
$$- \frac{21{,}421}{612{,}141{,}052{,}723{,}200}\lambda_3\lambda_4$$
$$+ \frac{11{,}747}{194{,}772{,}153{,}139{,}200}\lambda_1\lambda_2\lambda_4$$

- Etc.

$g=9$: degree 12, 10 terms.

$g=10$: degree 18, 21 terms.