25/03/26

Projective configuration counts (Cambridge)

(Joint w/ A. Fink and R. Silversmith)

§1: Cross-ratio degrees

Fix $n \geq 3$ and subsets

$$\boxed{I_1, \dots, I_{n-3} \in \binom{[n]}{4}}$$

Consider the forgetful maps:

$$M_{0,n} \xrightarrow{F_I} \prod_{j=1}^{n-3} M_{0,I_j}$$

Define the cross-ratio degree:

$$d_I := \deg F_I \in \mathbb{N}$$

Number of n-tuples $p_1, \dots, p_n \in \mathbb{P}^1$ up to PGL_2, satisfying $n-3$ cross-ratio constraints.

Connections to:

- Hypertree projections [Castravet-Tevelev '13]
- Linear GW theory [Silversmith '22]
- Scattering amplitudes [Silversmith '24]
- Graph rigidity [Gallet-Grasegger-Schicho '20]

Rich Combinatorial structure.

Illustrative example: the surplus.

We say $\mathcal{I} = (I_1 \_ I_{n-3})$ satisfies the surplus condition iff for all nonempty subsets $J \subseteq [n-3]$ we have:

$$\left| \bigcup_{j \in J} I_j \right| \geq |J| + 3$$

Thm [Jordán-Kaszanitzky, Brakensiek-Eur-Larson-Li]:

$\boxed{d_{\mathcal{I}} \neq 0 \text{ iff the surplus condition holds.}}$

Proof: Prove directly that $F_{\mathcal{I}}$ is dominant by studying its derivative.

Serious algebra: uses the "GM-MDS theorem" from the theory of error-correcting codes. □

Can extend to compactification:

$$\overline{M}_{0,n} \xrightarrow{F_{\mathcal{I}}} \prod_{j=1}^{n-3} \overline{M}_{0,I_j}$$

Then move chosen point downstairs into the boundary. Get either:

$$d_I = \int_{\overline{M}_{0,n}} \prod_{j=1}^{n-3} F^{x}_{I_j}(\psi)$$

$$= \int_{\overline{M}_{0,n}} \prod_{j=1}^{n-3} F^{x}_{I_j}(D_j)$$

This gives algorithm to compute.

Two key properties of $\overline{M}_{0,n}$:

(i) SNC compactification

$\Rightarrow$ boundary intersection theory (essentially) combinatorial

(ii) Modular

$\Rightarrow$ boundary combinatorics understandable

Remember this: these properties will be in tension later on.

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§2: Projective configuration counts

This was points on $\mathbb{P}^1$ up to PGL_2.

What about points on $\mathbb{P}^{r-1}$ up to PGL_r?

Generalise points being distinct to points being linearly general.

$P_1, \ldots, P_n \in \mathbb{P}^{r-1}$ linearly general if:

$$\left.\begin{array}{l} \bullet \text{ No 2 coincide} \\ \bullet \text{ No 3 on a line.} \\ \bullet \text{ No 4 on a plane} \\ \text{etc.} \end{array}\right\}$$

Fact: PGL_r acts simply-$(r+1)$ transitively on linearly general configurations in $\mathbb{P}^{r-1}$.

Thus $\exists$ unique representative of the PGL_r orbit with:

$P_1 = [1, 0, \ldots, 0]$, $P_2 = [0, 1, 0, \ldots, 0]$, $\ldots$, $P_r = [0, \ldots, 0, 1]$, $P_{r+1} = [1, 1, \ldots, 1]$.

Def^n: write $\boxed{X(r,[n])}$ for space of n labelled linearly general points in $\mathbb{P}^{r-1}$ up to PGL_r:

$$X(r,[n]) = \left((\mathbb{P}^{r-1})^n \right)^{\circ} / PGL_r.$$

Then we have:

$X(r,[r+1]) = pt$ (analogue of $M_{0,3}$)

$X(r,[r+2]) \underset{\text{open}}{\subseteq} \mathbb{P}^{r-1}$ (analogue of $M_{0,4}$)

$$\dim X(r,[n]) = (r-1)(n-r-1).$$

Fix $\boxed{I_1, \ldots, I_{n-r-1} \in \binom{[n]}{r+2}}$.

$$X(r,[n]) \xrightarrow{F_I} \prod_{j=1}^{n-r-1} X(r, I_j)$$

Define the projective configuration count:

$d_I := \deg F_I \in \mathbb{N}$.

Also have a rich combinatorial structure.

Much harder to study (will explain why at the end).

E.g. we have found an analogue of the nonvanishing criterion, and we have some evidence for it, but we can't prove it.

I'll tell you what we can do thus far.

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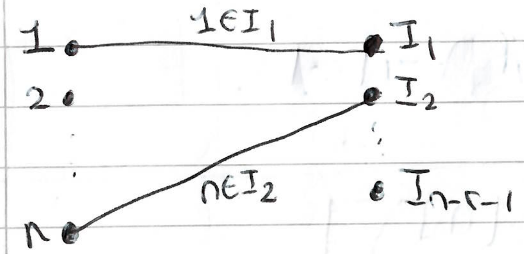
§3: Results: upper bound and dim. reduction

We produce a combinatorial upper bound.

Setup: $I_1, \ldots, I_{n-r-1} \in \binom{[n]}{r+2}$.

Record memberships $i \in I_j$ in bipartite graph:

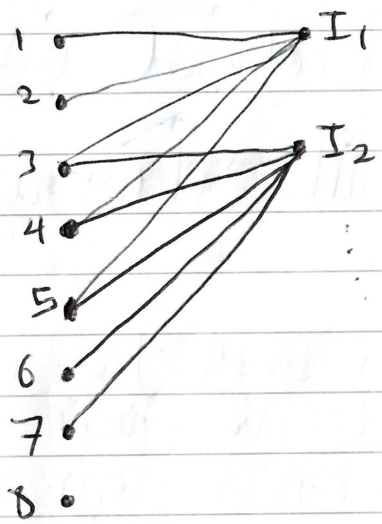


$\boxed{\Gamma(\mathcal{I})}$ configuration graph.

E.g: $r=3$ (so $\mathbb{P}^2$), $n=8$, so $n-r-1=4$. Take:

$$\mathcal{I} = (\underset{I_1}{12345}, \underset{I_2}{34567}, \underset{I_3}{56781}, \underset{I_4}{78123}).$$

Start of graph is:



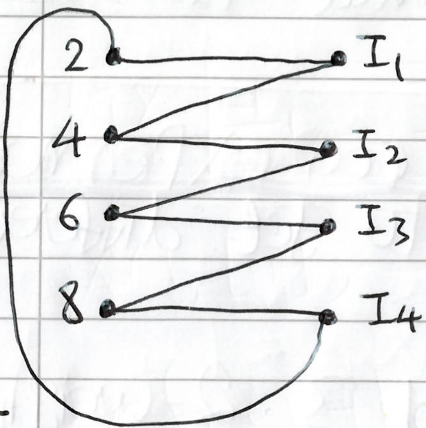
$\Gamma(\mathcal{I})$ has more left than right vertices.

Let's prune it: Choose

$$S \in \binom{[n]}{r+1}$$

and delete the left vertices indexed by S, producing $\Gamma(\mathcal{I}) \setminus S$.

E.g: Take $S = 1357$. Get:



$\boxed{T(\mathcal{I})\setminus S}$ pruned configuration graph.

Get bipartite graph with same number of left and right vertices.

Defn: Fix $m \geq 1$. An m-weighted transversal of T is an assignment of weights

$$m_e \in \mathbb{N}$$

to the edges $e \in E(T)$, such that for every $v \in V(T)$:

$$\sum_{e \ni v} m_e = m.$$

Note a 1-weighted transversal is just a perfect matching.

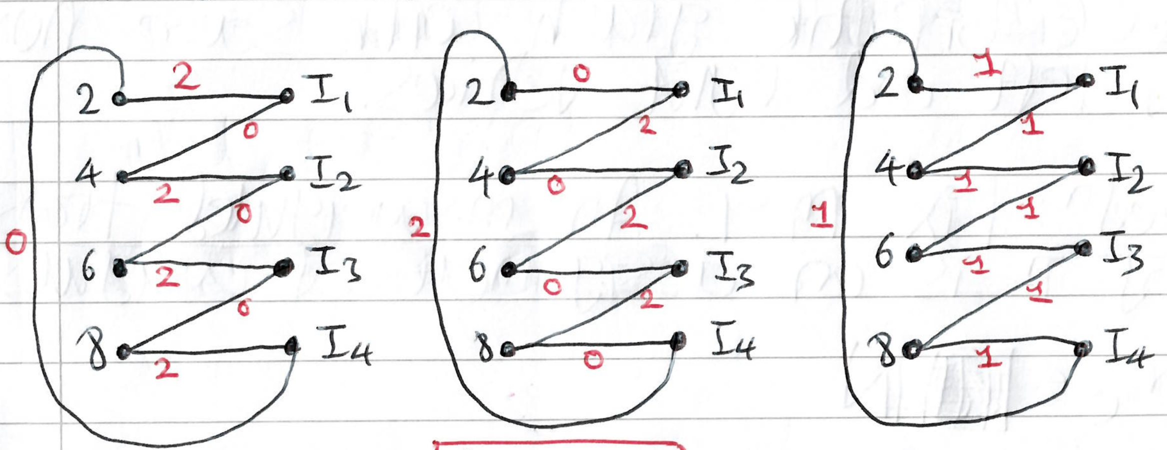
~~Thm [Fink-N-Silversmith]: $d-1$ is k the number of $(n-1)$-~~

Generalizes [Silversmith '22] for $r=2$.

Th^{m}**1 [Fink-N-Silversmith]:** $d_{\mathcal{I}}$ is $\leq$ the number of $(r-1)$-weighted transversals of $T(\mathcal{I})\setminus S$.

(For any choice of S: can experiment with different choices to get different upper bounds.)

E.g: $T(\mathcal{I})\setminus S$ has exactly 3 2-weighted transversals.



Therefore $\boxed{d_{\mathcal{I}} \leq 3}$.

In this case in fact $d_{\mathcal{I}}=3$ (Stochastic calculation).

Proof: Instead of counting configurations $(P_1, \dots, P_n)$, we can recast as counting elements

$g_1, \dots, g_k \in PGL_r$.

This allows us to write:

$$d_{\mathcal{I}} = \# \bigcap_{i=1}^{n} \Lambda_i \subseteq (PGL_r)^k$$

where $\boxed{\Lambda_i} \subseteq (PGL_r)^k$ are certain subvarieties.

(They have codim $= (r-1)(|J_i|-1)$.)

Then the key idea: compactify naively:

$(PGL_r)^k \subseteq (\mathbb{P}^{r^2-1})^k$.

Positivity of refined intersection products [Fulton Chapter 12] gives:

$$d_{\mathcal{I}} \leq \prod_{i=1}^{n} [\Lambda_i]$$

Remains to compute

$[\Lambda_i] \in A^*((\mathbb{P}^{r^2-1})^k)$

~~[illegible]~~, and equate product with number of $(r-1)$-weighted transversals.

Computing class uses Thom-Porteous and Jacobi-Trudi. $\square$

~~[illegible]~~

Similar strategy used by [Simms '26] to get upper bound for log GW invariants of $\mathbb{P}^m$.

Second result is dimension reduction.

There are projection maps

$X(r+1, [n+1]) \to X(r, [n])$

Projecting from the $(n+1)$st point onto the orthogonal hyperplane.

Th^m 2 [Fink-N-Silversmith]: Fix $\mathcal{I} = (I_1, \_, I_{n-r-1})$

Define:

$\mathcal{I}' = (I_1 \sqcup \{n+1\}, \_, I_{n-r-1} \sqcup \{n+1\})$.

Then we have:

$$d_{\mathcal{I}'} = d_{\mathcal{I}}$$

(in $X(r+1, [n+1])$) (in $X(r, [n])$)

Proof: Looks simple. Very hard to prove. Boils down to:

$$\begin{tikzcd} X(r+1, [n+1]) \arrow[r, "\varphi"] & F \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \prod_{j=1}^{n-r-1} X(r+1, I_j') \arrow[d] \\ & X(r, [n]) \arrow[r] & \prod_{j=1}^{n-r-1} X(r, I_j) \end{tikzcd}$$

Just need to show φ birational.

Devilish. Elite linear algebra. $\square$

§4: Limitations

For cross-ratio degrees a key technique was passing to the compactification $\overline{M}_{0,n}$ and pushing the intersection problem into the boundary.

The key properties of $\overline{M}_{0,n}$ were: snc and modular.

For $r \geq 3$, various (essentially) modular compactifications exist:

$$X(r,n) \subseteq \boxed{\overline{X}(r,n)}.$$

However, none are snc.

- KSBA:
  - Modular but has excess cpts.
  - Strata indexed by matroid tilings of hypersimplex $\Delta(r,n)$.
- Kapranov:
  - Chow quotient
  - Irreducible but badly singular (non-toroidal) at boundary.
  - Main cpt of KSBA, and in fact a union of strata: those indexed by regular matroid tilings.

- Schaffler-Tevelev:
  - Also defined as closure.
  - Motivation from Mustafin varieties [Gemitza-Piwek].
  - Irreducible but badly singular.
  - Partial resolution of Kapranov.

There's a reason none of the modular compactifications are snc!

Recall we can produce $\overline{M}_{0,n}$ as an iterated blowup:

$\overline{M}_{0,n} \longrightarrow (\mathbb{P}^1)^{n-3}$

can view as a Hassett space $\overline{M}_{0,w}$ for $w = (1,1,1,\varepsilon,\ldots,\varepsilon)$.

The blowup centres are precisely matroidal loci: loci where the points $p_1,\ldots,p_n$ satisfy some specified linear dependencies.

(In this $r=2$ case, the only linear dependencies are incidences, but for $r \geq 3$ we will need to consider collinearity, coplanarity etc., hence the language of matroids to encode this.)

For $r \geq 3$ can try to do the same, and construct $\overline{X}(r,[n])$ as an iterated blowup:

$$\overline{X}(r,[n]) \longrightarrow (\mathbb{P}^{r-1})^{n-r-1}$$

But here's the rub: this should be an iterated blowup along matroidal loci. But for $r \geq 3$ matroidal loci are awful:

(i) They are arbitrarily singular. [Mnëv]

(ii) They do not form an orderly stratification (closure of one is not a union of others).

This makes the iterated blowup hard to even construct, and unlikely to be smooth at the end.

It is helpful to have this point of view when thinking about the bad behaviour of existing compactifications.

But for the projective configuration counts not all is lost: there may be some tricks...