Recursion Formulae in Logarithmic Gromov–Witten Theory and Quasimap Theory

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For Omid.

I miss you.

Abstract

The primary theme of this thesis is the study of recursion formulae in enumerative geometry. In Chapter 2 we define moduli spaces of relative stable quasimaps in genus zero, and derive a recursion formula which allows us to compute the resulting relative quasimap invariants. We apply this formula to obtain a quantum Lefschetz theorem for quasimap invariants (this is joint work with Luca Battistella). In Chapter 3 we present work in progress towards a recursion formula for log Gromov–Witten invariants. Along the way, we introduce auxiliary moduli spaces and use them to probe the geometry of the moduli space of log stable maps. Finally in Chapter 4, we express a fundamental object in ordinary Gromov–Witten theory – Givental's Lagrangian cone – using relative stable maps. As a corollary, we obtain a sequence of universal relations involving the Gromov–Witten invariants.

Statement of originality

This thesis represents my own work. Any collaborations with or contributions from others are explicitly acknowledged in the text, in the form of references to published works or personal communications.

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CHAPTER 1

Introduction

1.1. Gromov–Witten theory

Gromov–Witten theory is a branch of enumerative geometry which studies parametrised curves in a fixed ambient variety X (assumed to be smooth). These parametrised curves are called *stable maps*, and consist of a nodal marked source curve C together with a map $f\colon C \to X$ which is stable in the sense that it has only finitely many automorphisms. These objects form a moduli space, denoted

$$\overline{\mathcal{M}}_{g,n}(X,\beta)$$

where $g \geq 0$ is the arithmetic genus of the source curve, $n \geq 0$ is the number of markings on the source curve and $\beta = f_*[C] \in \mathrm{H}_2^+(X)$ is the degree of the map. This moduli space is a proper Deligne–Mumford stack of finite type. It has a well-defined expected (or virtual) dimension, but in general it will contain components of excess dimension. On the other hand, it admits a natural perfect obstruction theory [BF97] [LT98] and so carries a virtual fundamental class

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}} \in \mathrm{A}_{\mathrm{vdim}}\,\overline{\mathcal{M}}_{g,n}(X,\beta)$$

which can be thought of as the fundamental class of a suitably generic perturbation of the moduli space. For each marked point $x_i \in \{x_1,\ldots,x_n\}$ there is an evaluation map

$$\mathrm{ev}_i\colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to X$$

and if we are given cohomology classes $\gamma_1,\ldots,\gamma_n \in \mathrm{H}^*(X)$ such that

$$\sum_{i=1}^{n} \mathrm{codim}(\gamma_i) = \mathrm{vdim}\,\overline{\mathcal{M}}_{g,n}(X,\beta)$$

then we can define the corresponding Gromov–Witten invariant:

$$\langle \gamma_1,\ldots,\gamma_n\rangle^X_{g,n,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}}} \prod_{i=1}^{n} \mathrm{ev}_i^*\,\gamma_i$$

This should be thought of as giving a (virtual) count of the number of genus g, degree β curves in X which pass through γ_i at the point x_i. Good introductions to Gromov–Witten theory are given in [FP97] [Gat03a] and [CK99, §§7-9]. The original references are [Kon95] [Beh99] [BM96].

1.2. Relative Gromov–Witten theory

Relative Gromov–Witten theory enhances the above picture by introducing a smooth hypersurface Y in X. In addition to the numerical data g, n and β we also fix a vector

$$\alpha = (\alpha_1, \ldots, \alpha_n)$$

of non-negative integers such that $\Sigma_{i=1}^n \alpha_i = Y \cdot \beta$. We are then interested in studying stable maps to X which have tangency α_i to Y at the marked point x_i. These will be called *relative stable maps.*

There are a number of different ways to define these objects: the intersection-theoretic approach of A. Gathmann [Gat02], the expanded degenerations approach of J. Li [Li01, Li02], the orbifold expanded degenerations approach of D. Abramovich and B. Fantechi [AF16], the logarithmic expanded degenerations approach of B. Kim [Kim10] and, most recently, the logarithmic (non-expanded) approach of D. Abramovich, Q. Chen, M. Gross and B. Siebert [GS13, Che14, AC14]. In each case, we obtain a moduli space of relative stable maps

$$\overline{\mathcal{M}}_{g,\alpha}(X|Y,\beta)$$

which enjoys all the good properties of the moduli space of (ordinary) stable maps: it is a proper Deligne–Mumford stack of finite type, with a well-defined virtual dimension and accompanying virtual fundamental class. In exactly the same manner as before, we obtain enumerative invariants, called relative Gromov–Witten invariants.

Relative Gromov–Witten invariants are important for several reasons: they can be used as a tool for calculating ordinary Gromov–Witten invariants [Gat03b] [MP06]; they can be used to study other moduli spaces, for instance the moduli space of curves [GV05]; and they play a fundamental role in the theory of intrinsic Mirror Symmetry [GS16], an important area of modern algebraic geometry with close connections to theoretical physics.

1.3. Recursion formulae

The primary theme of this work is the study of recursion formulae in relative enumerative geometry. In [Gat02], Gathmann establishes a recursion formula which allows one to compute relative Gromov–Witten invariants in genus zero by repeatedly "decreasing the tangencies" of the marked points to the divisor. This formula has numerous applications, including a new proof of the Mirror Theorem [Gat03b]. In this thesis, we extend Gathmann's formula to other settings, and use these extensions to obtain new results. In particular, we focus on applications to quasimap theory and logarithmic Gromov–Witten theory.

In the quasimap setting, we define relative quasimap invariants in genus zero and obtain a recursion formula which can be used to compute them. We then apply this recursion to prove a quantum Lefschetz theorem for quasimap invariants; a result which, on the face of it, has nothing to do with relative quasimaps. This is joint work with Luca Battistella.

In the logarithmic setting, we present work in progress towards obtaining a version of Gathmann's formula for log Gromov–Witten invariants relative an snc divisor. In the process, we are led to define certain auxiliary moduli spaces of stable maps to an snc divisor, and to compare these to the ordinary moduli spaces of log stable maps. This leads to interesting insights into the geometry of the latter moduli spaces. While the full recursion formula is still work in progress, the incomplete version we currently have is sufficient to compute a large number of log Gromov–Witten invariants, which we illustrate through a series of examples.

1.4. Relative quantisation formalism

There is a secondary theme lurking in the background here, guided by the following (somewhat vague) question: what is the overarching structure governing the relative Gromov–Witten invariants? In the absolute setting, one answer to this question is given by A. Givental's quantisation formalism [Giv01a]. This has proven to be an extremely powerful framework for organising and proving results; see for instance [CI14] [CIJ14]. It would be of great interest to have a similarly powerful formalism in the relative setting. The first step towards such a theory would be a relative version of the Mirror Theorem. Since the classical Mirror Theorem can be interpreted as a wall-crossing phenomenon between Gromov–Witten invariants and quasimap invariants [CFK14], our work on relative quasimaps opens the way for a relative Mirror Theorem.

Once a relative Mirror Theorem has been established, the next step will be to obtain a relative version of Givental's Lagrangian cone. In the final chapter of this thesis, we show how the (ordinary) Lagrangian cone can be constructed using relative stable maps. This provides a strong hint as to how one might go about constructing a Lagrangian cone in the relative setting.

1.5. Outline

This thesis consists of three main chapters:

- Chapter 2. Here we discuss our joint work with Luca Battistella, where we define moduli spaces of relative quasimaps (in genus zero), establish a recursion formula for them and apply this recursion to obtain a quantum Lefschetz theorem for quasimap invariants.
- Chapter 3. Here we discuss work in progress towards a Gathmann-like recursion formula for log Gromov–Witten invariants of snc divisors. Along the way, we define auxiliary moduli spaces and use them to probe the geometry of the moduli space of log stable maps. We also include an appendix, giving a brief introduction to log geometry.
- Chapter 4. Here we give an interpretation of Givental's Lagrangian cone in terms of relative stable maps, and apply this to obtain universal relations involving the Gromov–Witten invariants.

Since these chapters are more or less logically independent, we have opted to provide each chapter with its own separate introduction.

1.6. Acknowledgements

Many, many people have contributed to making this thesis a reality. First and foremost, I would like to express my heartfelt thanks to my supervisor, Tom Coates, for years of support and encouragement and, most of all, for believing in me when I wasn't able to.

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CHAPTER 2

Quasimap quantum Lefschetz via relative quasimaps

The entirety of this chapter is joint work with Luca Battistella, first appearing as [BN17].

Abstract. We define moduli spaces of relative toric quasimaps in genus zero, in the spirit of A. Gathmann. When X is a smooth toric variety and Y is a very ample hypersurface in X we construct a virtual class on the moduli space of relative quasimaps to (X, Y) which can be used to define relative quasimap invariants of the pair. We obtain a recursion formula which expresses each relative invariant in terms of invariants of lower multiplicity. Finally we apply this formula to derive a quantum Lefschetz theorem expressing the restricted quasimap invariants of Y in terms of those of X. We include appendices collecting proofs of standard results in quasimap theory.

2.1. Introduction

In this chapter we construct moduli spaces of relative quasimaps as substacks of moduli spaces of (absolute) quasimaps. This provides a common generalisation of two different theories: stable quasimaps on the one hand, and relative stable maps (in the sense of A. Gathmann) on the other. In this introductory section we briefly recall these, putting our work in its proper context.

2.1.1. **Stable quasimaps**. The moduli space of *stable toric quasimaps* $\mathcal{Q}_{g,n}(X, \beta)$ was constructed by I. Ciocan-Fontanine and B. Kim [CFK10] as a compactification of the moduli space of smooth curves in a smooth and complete toric variety X. Roughly speaking, the objects are rational maps $C \dashrightarrow X$ where C is a nodal curve, subject to a stability condition; the precise definition depends on the description of X as a GIT quotient. The space $\mathcal{Q}_{g,n}(X, \beta)$ is a proper Deligne–Mumford stack of finite type. It admits a virtual fundamental class, which is used to define curve-counting invariants for X called *quasimap invariants.*

This theory agrees with that of stable quotients [MOP11] when both are defined, namely when X is a projective space. There is a common generalisation given by the theory of stable quasimaps to GIT quotients [CFKM14]. For simplicity, however, we will work mostly in the toric setting[1]. Thus in this chapter when we say "quasimaps" we are implicitly talking about toric quasimaps. Quasimap invariants provide an alternative system of curve counts to the more well-known Gromov–Witten invariants. These latter invariants are defined via moduli spaces of stable maps, and as such we will often refer to them as *stable map invariants*.

For X sufficiently positive, the quasimap invariants coincide with the Gromov–Witten invariants, in all genera. This has been proven in the following cases:

- X a projective space or a Grassmannian: see [MOP11, Theorems 3 and 4], and [Man14] for an alternative proof.
- X a projective complete intersection of Fano index at least 2: see [CFK16, Corollary 1.7], and [CZ14] for an earlier approach.
- X a projective toric Fano variety: see [CFK17, Corollary 1.3].

In general, however, the invariants differ, the difference being encoded by certain wall-crossing formulae, which can be interpreted in the context of toric mirror symmetry [CFK14] [CFK16].

2.1.2. **Relative stable maps**. Let Y be a smooth very ample hypersurface in a smooth projective variety X. In [Gat02] A. Gathmann constructs a moduli space of relative stable maps to the pair (X, Y) as a closed substack of the moduli space of (absolute) stable maps to X:

$$\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta) \hookrightarrow \overline{\mathcal{M}}_{0,n}(X,\beta)$$

The relative moduli space parametrises stable maps with prescribed tangencies to Y at the marked points. Unfortunately this space does not admit a natural perfect obstruction theory. Nevertheless, because Y is very ample it is still possible to construct a virtual fundamental class by intersection-theoretic methods, and hence one can define relative stable map invariants.

Gathmann establishes a recursion formula for these virtual classes which allows one to express any relative invariant of (X, Y) in terms of absolute invariants of Y and relative invariants with lower contact multiplicities. By successively increasing the contact multiplicites from zero to the maximum possible value, this gives an algorithm expressing the (restricted) invariants of Y in terms of those of X: see [Gat02, Corollary 5.7]. In [Gat03b] this result is applied to give an alternative proof of the mirror theorem for projective hypersurfaces [Giv96].

[1]This restriction is not essential for our arguments; the case of GIT quotients will be addressed in forthcoming work.

2.1.3. **Relative stable quasimaps**. In this chapter we construct moduli spaces of relative quasimaps in genus zero. We prove a recursion relation similar to Gathmann's formula, and use this to derive a quantum Lefschetz formula for quasimap invariants. Our construction carries over to the setting of ϵ-stable quasimaps [CFK10]; since for $\epsilon > 1$ these moduli spaces agree with the space of stable maps, one can view our construction as giving a common generalisation of the two stories outlined above.

The plan of the chapter is as follows. In §§2.2.1 and 2.2.2 we provide a brief review of the theories of stable quasimaps and relative stable maps. Then in §2.2.3 we define the moduli space of relative quasimaps as a substack of the moduli space of (absolute) quasimaps:

$$\mathcal{Q}_{0,\alpha}(X|Y,\beta) \hookrightarrow \mathcal{Q}_{0,n}(X,\beta).$$

Here X is a smooth toric variety, Y is a smooth very ample hypersurface and $\alpha = (\alpha_1, \ldots, \alpha_n)$ encodes the orders of tangency of the marked points to Y. Note that we *do not* require Y to be toric.

In §2.3 we examine the special case of a hyperplane $H \subseteq \mathbb{P}^N$. We find that the moduli space is irreducible of the expected dimension (in fact, more than this: it is the closure of the so-called "nice locus" consisting of maps from a $\mathbb{P}^1$ whose image is not contained in H). Thus it has an actual fundamental class, which we can use to define relative quasimap invariants. Another useful fact about this special case is that there exists a birational comparison morphism:

$$\chi : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N, d) \to \mathcal{Q}_{0,n}(\mathbb{P}^N, d)$$

This restricts to a birational morphism between the relative spaces, which we use to push down Gathmann's formula to obtain a recursion formula for relative stable quasimaps.

In §2.4 we turn to the case of an arbitrary pair (X,Y) with Y very ample. We use the embedding $X \hookrightarrow \mathbb{P}^N$ defined by $\mathcal{O}_X(Y)$ to construct a virtual class $[\mathcal{Q}_{0,\alpha}(X|Y,\beta)]^{\text{virt}}$. We then prove the recursion formula for (X,Y) by pulling back the formula for $(\mathbb{P}^N, H)$. This requires several comparison theorems for virtual classes, extending results in Gromov–Witten theory to the setting of quasimaps. The full statement of the recursion formula is:

Theorem 2.4.1. Let X be a smooth and proper toric variety and let $Y \subseteq X$ be a very ample hypersurface (not necessarily toric). Then

$$(\alpha_k \psi_k + \text{ev}_k^*[Y]) \cap [\mathcal{Q}_{0,\alpha}(X|Y,\beta)]^{\text{virt}} = [\mathcal{Q}_{0,\alpha+e_k}(X|Y,\beta)]^{\text{virt}} + [\mathcal{D}^{\mathcal{Q}}_{\alpha,k}(X|Y,\beta)]^{\text{virt}}$$

in the Chow group of $\mathcal{Q}_{0,\alpha}(X|Y,\beta)$.

Here $\mathcal{D}^{\mathcal{Q}}_{\alpha,k}(X|Y,\beta)$ is a certain ***quasimap comb locus*** sitting inside the boundary of the relative space (see §2.4.3); its virtual class should be thought of as a correction term. Such terms also appear in Gathmann's stable map recursion formula; however, in our setting the stronger stability condition for quasimaps considerably reduces the number of such contributions.

Finally in §2.5 we apply the recursion formula of §2.4 to obtain a quantum Lefschetz theorem for quasimap invariants. This takes two forms: first we have a general result which holds without any special restrictions on Y.

Theorem 2.5.1. Let X be a smooth projective toric variety and $Y \subseteq X$ a smooth very ample hypersurface. Then there is an explicit algorithm to recover the (restricted) quasimap invariants of Y from the quasimap invariants of X.

If, however, we are willing to impose some (semi)positivity assumptions, we can do better and actually obtain a closed formula (rather than just an algorithm) for this relationship.

Theorem 2.5.2. Let X be a smooth toric Fano variety and let $i\colon Y \hookrightarrow X$ be a very ample hypersurface. Assume that $-K_Y$ is nef and that Y contains all curve classes (see §2.5.3). Then

$$\frac{\sum_{\beta\geq 0} q^{\beta} \prod_{j=0}^{Y\cdot\beta}(Y+jz)S_0^X(z,\beta)}{P_0^X(q)} = \tilde{S}_0^Y(z,q)$$

where $S_0^X(z,\beta)$ and $\tilde{S}_0^Y(z,q)$ are the following generating functions for 2-pointed quasimap invariants

$$S_0^X(z,\beta) = (\mathrm{ev}_1)_* \left(\frac{1}{z-\psi_1}[\mathcal{Q}_{0,2}(X,\beta)]^{\mathrm{virt}}\right)$$

$$\tilde{S}_0^Y(z,q) = i_* \sum_{\beta\geq 0} q^{\beta} (\mathrm{ev}_1)_* \left(\frac{1}{z-\psi_1}[\mathcal{Q}_{0,2}(Y,\beta)]^{\mathrm{virt}}\right)$$

and $P_0^X(q)$ is given by:

$$P_0^X(q) = 1 + \sum_{\substack{\beta>0 \\ K_Y\cdot\beta=0}} q^{\beta}(Y\cdot\beta)!\langle[\mathrm{pt}_X]\psi_1^{Y\cdot\beta-1}, \mathbb{1}_X\rangle^X_{0,2,\beta}$$

The argument is similar in spirit to the one given in [Gat03b], however the stronger stability condition considerably simplifies both the proof and the final result. This formula can also be obtained as a consequence of [CFK14, Corollary 5.5.1]; see §2.5.6.

We also include two appendices, collecting together results which are well-known to experts but absent from the literature. Appendix 2.6 contains foundational results in quasimap theory, including functoriality and the splitting axiom, while Appendix 2.7 contains a number of intersection-theoretic lemmas used in the main body.

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2.1.5. **Table of notation**. We will use the following notation, most of which is introduced in the main body of the chapter.

X	a smooth projective toric variety
Y	a smooth very ample hypersurface in X
Σ	the fan of X
$\Sigma(1)$	the set of 1-dimensional cones of Σ
ρ	an element of $\Sigma(1)$
D_ρ	the toric divisor in X corresponding to ρ
$\overline{\mathcal{M}}_{g,n}(X,\beta)$	the moduli space of stable maps to X
$\overline{\mathcal{M}}_{0,\alpha}(X\|Y,\beta)$	the moduli space of relative stable maps to (X,Y); see §2.2.2
$\mathcal{Q}_{g,n}(X,\beta)$	the moduli space of toric quasimaps to X; see §2.2.1
$\mathcal{Q}^\circ_{0,\alpha}(X\|Y,\beta)$	the nice locus of relative quasimaps to (X,Y); see §2.3.1
$\mathcal{Q}_{0,\alpha}(X\|Y,\beta)$	the moduli space of relative quasimaps to (X,Y); see §2.2.3
$\mathcal{D}^{\mathcal{Q}}_{\alpha,k}(X\|Y,\beta)$	the quasimap comb locus; see §2.3.2
$\mathcal{D}^{\mathcal{Q}}(X\|Y,A,B,M)$	(a component of) the comb locus; see §2.3.2
$\mathcal{E}^{\mathcal{Q}}(X\|Y,A,B,M)$	the total product for the comb locus; see §2.4.3
$\mathcal{D}^{\mathcal{Q}}(X,A,B)$	the quasimap centipede locus; see Appendix 2.6.3
$\mathcal{E}^{\mathcal{Q}}(X,A,B)$	the total product for the centipede locus; see Appendix 2.6.3

$\mathfrak{M}^{\mathrm{wt}}_{g,n}$	the moduli stack of weighted prestable curves; see Appendix 2.6.3
$\mathfrak{Pic}^{d,\mathrm{st}}_{g,n}$	an open substack of the relative Picard stack of the universal curve over $\mathfrak{M}_{g,n}$
$\mathfrak{Bun}^{g,n}_{G}$	the moduli stack of principal G-bundles on the universal curve over $\mathfrak{M}_{g,n}$; see Appendix 2.6.4
$\mathcal{Q}(f)$	the push-forward morphism between quasimap spaces; see Appendix 2.6.1
χ	the comparison morphism from stable maps to quasimaps; see §2.3.1
$f^!$	Gysin morphism for f a regular embedding
$f^!_{\mathrm{v}}$	virtual pull-back for f virtually smooth; see Appendix 2.7
$f^!_{\Delta}$	diagonal pull-back; see Appendix 2.7

2.2. Relative stable quasimaps

2.2.1. **Review of absolute stable quasimaps**. We briefly recall the definition and basic properties of the moduli space of toric quasimaps; see [CFK10] for more details.

Definition 2.2.1. [CFK10, Definition 3.1.1] Let N be a lattice, let $\Sigma \subseteq N_{\mathbb{Q}}$ be a fan, and let $X = X_\Sigma$ be the corresponding toric variety. Suppose that X is smooth and projective. Let $M = N^\vee = \mathrm{Hom}(N, \mathbb{Z})$ and let $\mathcal{O}_{X_\Sigma}(1)$ be a fixed polarisation, which we can write (non-uniquely) in terms of the torus-invariant divisors as:

$$\mathcal{O}_{X_\Sigma}(1) = \otimes_{\rho\in\Sigma(1)}\mathcal{O}_{X_\Sigma}(D_\rho)^{\otimes\alpha_\rho}$$

for some $\alpha_\rho \in \mathbb{Z}$. We fix the following numerical invariants: a genus $g \geq 0$, a number of marked points $n \geq 0$, and an effective curve class $\beta \in \mathrm{H}_2^+(X)$. A *stable (toric) quasimap* is given by the data

$$\Big((C, x_1, \ldots, x_n), (L_\rho, u_\rho)_{\rho\in\Sigma(1)}, (\varphi_m)_{m\in M}\Big)$$

where:

(1) $(C, x_1, \ldots, x_n)$ is a prestable curve of genus g with n marked points;
(2) the L_ρ are line bundles on C of degree $d_\rho = D_\rho \cdot \beta$;
(3) the u_ρ are global sections of L_ρ;
(4) $\varphi_m \colon \bigotimes_{\rho\in\Sigma(1)} L_\rho^{\otimes\langle\rho,m\rangle} \to \mathcal{O}_C$ are isomorphisms, such that $\varphi_m \otimes \varphi_{m'} = \varphi_{m+m'}$ for all $m, m' \in M$.

These are required to satisfy the following two conditions:

(1) *nondegeneracy*: there is a finite (possibly empty) set of smooth and non-marked points $B \subseteq C$, called the *basepoints* of the quasimap, such that for all $x \in C \setminus B$ there exists a maximal cone $\sigma \in \Sigma_{\max}$ with $u_\rho(x) \neq 0$ for all $\rho \not\subset \sigma$;

(2) *stability*: if we let $L = \otimes_\rho L_\rho^{\otimes \alpha_\rho}$ then the following $\mathbb{Q}$-divisor is ample

$$\omega_C(x_1 + \ldots + x_n) \otimes L^{\otimes \epsilon}$$

for every rational $\epsilon > 0$. This does not depend on the choice of polarisation. Note that necessarily $2g - 2 + n \geq 0$.

Remark 2.2.2. This definition is motivated by D. A. Cox's description of the functor of points of a toric variety in terms of Σ-collections [Cox95a]; see also Appendix 2.6.1. A quasimap defines[2] a rational map $C \dashrightarrow X$ with base locus equal to B. In particular a quasimap without any basepoints defines a morphism $C \to X$. Thus maps with basepoints appear in the (virtual) boundary of the moduli space of quasimaps, in much the same way as maps with rational tails appear in the boundary of the moduli space of stable maps. This is something more than just a vague analogy; these loci correspond to each other under the comparison morphism when $X = \mathbb{P}^N$; see §2.3.1.

More generally, one can define the notion of a family of quasimaps over a base scheme S, and what it means for two such families to be isomorphic; one thus obtains a moduli stack

$$\mathcal{Q}_{g,n}(X, \beta)$$

of stable (toric) quasimaps to X, which is a proper Deligne–Mumford stack of finite type [CFK10, §3].

As with the case of stable maps, there is a combinatorial characterisation of stability which is easy to check in practice; a prestable quasimap is stable if and only if the following conditions hold:

(1) the line bundle $L = \otimes_\rho L_\rho^{\otimes \alpha_\rho}$ must have strictly positive degree on any rational component with fewer than three special points, and on any elliptic component with no special points;

[2] This can be expressed in a more generalisable manner as follows: a quasimap is a map to the stack quotient $\left[\mathbb{A}^{\Sigma(1)} / \mathbb{G}_{\mathrm{m}}^r\right]$ such that B is the preimage of the unstable locus.

(2) C cannot have any rational components with fewer than two special points (that is, no *rational tails*).

Condition (1) is analogous to the ordinary stability condition for stable maps. Condition (2) is new, however, and gives quasimaps a distinctly different flavour to stable maps; we shall sometimes refer to it as the *strong stability condition.*

Remark 2.2.3. Unlike in Gromov–Witten theory, $\mathcal{Q}_{g,n+1}(X,\beta)$ is *not* the universal curve over $\mathcal{Q}_{g,n}(X,\beta)$ since markings cannot be basepoints. In fact there is not even a morphism between these spaces in general.

The moduli space $\mathcal{Q}_{g,n}(X,\beta)$ admits a perfect obstruction theory relative to the moduli space $\mathfrak{M}_{g,n}$ of source curves [CFK10, §5], and hence one can construct a virtual class

$$[\mathcal{Q}_{g,n}(X,\beta)]^{\mathrm{virt}} \in \mathrm{A}_{\mathrm{vdim}\,\mathcal{Q}_{g,n}(X,\beta)}\left(\mathcal{Q}_{g,n}(X,\beta)\right)$$

where the virtual dimension is the same as for stable maps:

$$\mathrm{vdim}\,\mathcal{Q}_{g,n}(X,\beta) = (\dim X - 3)(1-g) - (K_X \cdot \beta) + n$$

Since the markings are not basepoints there exist evaluation maps

$$\mathrm{ev}_i : \mathcal{Q}_{g,n}(X,\beta) \to X$$

and there are ψ-classes defined in the usual way by pulling back the relative dualising sheaf of the universal curve

$$\psi_i = \mathrm{c}_1(x_i^* \omega_{\mathcal{C}/\mathcal{Q}})$$

where $\mathcal{C} \to \mathcal{Q} = \mathcal{Q}_{g,n}(X,\beta)$ is the universal curve and $x_i : \mathcal{Q} \to \mathcal{C}$ is the section defining the ith marked point. Putting all these pieces together, we can define *quasimap invariants*:

$$\langle \gamma_1 \psi_1^{k_1}, \ldots, \gamma_n \psi_n^{k_n} \rangle^X_{g,n,\beta} = \int_{[\mathcal{Q}_{g,n}(X,\beta)]^{\mathrm{virt}}} \prod_{i=1}^n \mathrm{ev}_i^*(\gamma_i) \cdot \psi_i^{k_i}$$

We use the same correlator notation as in Gromov–Witten theory; this should not cause any confusion.

Example 2.2.4. Consider $\mathcal{Q}_{0,2}(\mathbb{P}^2, 1)$. What are its objects? By the strong stability condition (2) above, we see that the source curve must be irreducible. On the other hand since $\mathbb{P}^2$ has Picard rank 1 we may exploit the isomorphisms φ_m to reduce ourselves to considering one line bundle equipped with three sections. Thus the data of the quasimap is $((C, x_1, x_2), L, u_0, u_1, u_2)$ where $(C, L) \cong (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$.

Pick coordinates $[s : t]$ on $\mathbb{P}^1$ such that the marked points are $[1 : 0]$ and $[0 : 1]$. We can express the sections as $u_i = a_i s + b_i t$; the requirement that the markings are not basepoints then translates into the following stability condition:

$$(a_0, a_1, a_2) \neq (0, 0, 0) \quad \text{and} \quad (b_0, b_1, b_2) \neq (0, 0, 0).$$

The group $\mathrm{Aut}(C; x_1, x_2) \cong \mathbb{G}_{\mathrm{m}}$ acts by rotation $\lambda \colon [s : t] \mapsto [s : \lambda t]$, while $\mathrm{Aut}(L) \cong \mathbb{G}_{\mathrm{m}}$ acts by scalar multiplication on $\underline{a}$ and $\underline{b}$. Thus the $\mathbb{G}_{\mathrm{m}}^2$ action on $\mathbb{A}^6_{\underline{a},\underline{b}}$ is encoded by the following weight matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

It is now clear that the quotient is $\mathbb{P}^2 \times \mathbb{P}^2$; in fact, we see that the evaluation map

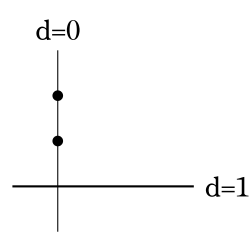
$$(\mathrm{ev}_1, \mathrm{ev}_2) \colon \mathcal{Q}_{0,2}(\mathbb{P}^2, 1) \to \mathbb{P}^2 \times \mathbb{P}^2$$

is an isomorphism. It is given in the above notation by:

$$((\mathbb{P}^1; [1 : 0], [0 : 1]); \mathcal{O}_{\mathbb{P}^1}(1); u_0, u_1, u_2) \mapsto ([a_0 : a_1 : a_2], [b_0 : b_1 : b_2])$$

Notice that the locus where $(a_0, a_1, a_2) = \mu(b_0, b_1, b_2)$, i.e. the diagonal in $\mathbb{P}^2 \times \mathbb{P}^2$ is precisely the locus of quasimaps which have a basepoint. The point $[a_0 : a_1 : a_2] = [b_0 : b_1 : b_2] \in \mathbb{P}^2$ is the image of the underlying "residual map" of degree 0, obtained by dividing all the sections by a local equation of the basepoint (equivalently, by extending the rational map $C \dashrightarrow \mathbb{P}^2$ to a morphism $C \to \mathbb{P}^2$).

On the other hand, $(\mathrm{ev}_1, \mathrm{ev}_2) \colon \overline{\mathcal{M}}_{0,2}(\mathbb{P}^2, 1) \to \mathbb{P}^2 \times \mathbb{P}^2$ is *not* an isomorphism. Off the diagonal, the images of the two marked points determine uniquely the image of the stable map, i.e. the line through them. On the diagonal however, the following maps with a rational tail appear:



The image of the degree 1 component under f can be any line passing through the point of $\mathbb{P}^2$ to which the other component is contracted. Hence $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^2, 1) \cong \mathrm{Bl}_\Delta(\mathbb{P}^2 \times \mathbb{P}^2)$. The comparison morphism χ (see §2.3.1) can be interpreted as the blow-down map, and it induces an isomorphism of the rational tail-free locus with the basepoint-free locus.

Remark 2.2.5. There is a more general notion of *ϵ-stable quasimap* [CFKM14, §7.1]. Here the stability condition, namely that the line bundle

$$\omega_C(x_1 + \ldots + x_n) \otimes L^{\otimes \epsilon}$$

is ample, is only required to hold for a fixed $\epsilon \in \mathbb{Q}_{>0}$ (instead of for arbitrary ϵ, as was the case with ordinary quasimaps).

This has the effect of allowing some rational tails to appear, as long as their degree is high enough with respect to ϵ. In order to keep the moduli space Deligne–Mumford and seperated, one also has to bound the multiplicity of the basepoints that can occur.

By boundedness and the fact that the degree is an integer-valued function, there exist finitely many critical values of ϵ which divide $\mathbb{Q}_{>0}$ into chambers inside which the moduli spaces $\mathcal{Q}^{\epsilon}_{g,n}(X, \beta)$ do not change. For ϵ sufficiently small we recover the space of (ordinary) quasimaps, and for ϵ sufficiently large we obtain the moduli space of stable maps. Thus one can view the spaces of ϵ-stable quasimaps as interpolating between these two extremes, and they have proven successful as a tool for comparing quasimap invariants to stable map invariants [Tod11] [CFK14].

Another variant of the theory, which will play a role in later sections, is that of *parametrised quasimaps* [CFK10, §7]. A parametrised quasimap comes with a preferred rational component, which is equipped with the extra data of an isomorphism with $\mathbb{P}^1$, and the stability condition is imposed *on all but the preferred component.* This mimics the construction of graph spaces in Gromov-Witten theory – for example, there is a $\mathbb{G}_{\mathrm{m}}$-action on $\mathcal{Q}G_{g,n}(X, \beta)$ by rotating the preferred component, which plays the role of the $\mathbb{G}_{\mathrm{m}}$-action that rotates the graph. The fixed loci and their equivariant normal bundles are well-understood, at least in the toric setting [CFK10, §7]. In the parametrised case we no longer require $2g - 2 + n \geq 0$, due to the modified stability condition. In particular it makes sense, and turns out to be very useful, to consider unmarked parametrised quasimaps $\mathcal{Q}G_{0,0}(X, \beta)$. In this case the source curve is necessarily irreducible.

Example 2.2.6. $\mathcal{Q}G_{0,0}(\mathbb{P}^N, d) = \mathbb{P}^k$ with $k = (N+1)(d+1) - 1$. Indeed, the curve and line bundle must be $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$ and we are left with choosing $N+1$ sections of $\mathcal{O}_{\mathbb{P}^1}(d)$ (not all zero) up to automorphisms of $\mathcal{O}_{\mathbb{P}^1}(d)$, i.e. up to scaling. For early appearances of such spaces, see for instance [Giv98] [MP95] [Ber00].

2.2.2. **Review of relative stable maps**. Given a smooth projective variety X and a smooth very ample divisor Y, Gathmann's moduli space of relative stable maps parametrises stable maps to X with specified tangencies to Y at the marked points.

Definition 2.2.7. [Gat02, Definition 1.1] Let X be a smooth projective variety and $Y \subseteq X$ a smooth very ample divisor. Fix a number $n \geq 0$ of marked points, an effective curve class $\beta \in \mathrm{H}_2^+(X)$ and an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers such that $\Sigma_i \alpha_i \leq Y \cdot \beta$. The moduli space

$$\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)$$

of relative stable maps to (X, Y) is defined to be the locus in $\overline{\mathcal{M}}_{0,n}(X, \beta)$ of stable maps $(C \to S, (x_i : S \to C)_{i=1}^n, f : C \to X)$ satisfying the following two conditions:

(1) if x_i is a marked point such that $\alpha_i > 0$ then $f(x_i) \in Y$;
(2) if we consider the class $f^*[Y] \in \mathrm{A}^0(f^{-1}Y \to S)$ then the difference $f^*[Y] - \sum_i \alpha_i[x_i]$ is an effective class.

These conditions define a closed substack of $\overline{\mathcal{M}}_{0,n}(X, \beta)$. Condition (1) is required in order for the class $\sum_i \alpha_i[x_i]$ to make sense in $\mathrm{A}^0(f^{-1}Y \to S)$.

Remark 2.2.8. The notation in (2) comes from bivariant intersection theory: see [Ful98, §17]. Fibrewise, the condition is that the class $f^*[Y] - \sum_i \alpha_i[x_i] \in A_0(f^{-1}Y)$ is required to be effective.

The definition given above works in families; however there is an equivalent, more combinatorial definition for individual maps which is more useful in practice (see [Gat02, Remark 1.4]): a stable map $(C, x_1, \ldots, x_n, f)$ is a relative stable map if and only if, for each connected component Z of $f^{-1}(Y) \subseteq C$:

(1) if Z is a point and is equal to a marked point x_i, then the multiplicity of f to Y at x_i is greater than or equal to α_i;
(2) if Z is one-dimensional (hence a union of irreducible components of C) and if we let $C^{(i)}$ for $1 \leq i \leq r$ denote the irreducible components of C adjacent to Z and $m^{(i)}$ denote the multiplicity of $f|_{C^{(i)}}$ to Y at the node $Z \cap C^{(i)}$, then:

$$(*) \qquad Y \cdot f_*[Z] + \sum_{i=1}^{r} m^{(i)} \geq \sum_{x_i \in Z} \alpha_i$$

Remark 2.2.9. In case (2) above we call Z an *internal* component and the $C^{(i)}$ *external* components. Note that Z is not necessarily irreducible.

Remark 2.2.10. When $\alpha = (0,\ldots,0)$, condition (2) becomes $Y \cdot \beta \geq 0$, so $\overline{\mathcal{M}}_{0,(0,\ldots,0)}(X|Y,\beta) = \overline{\mathcal{M}}_{0,n}(X,\beta)$ as long as Y is nef.

Remark 2.2.11. In the case of maximal multiplicity $\Sigma_i \alpha_i = Y \cdot \beta$, all the inequalities in the above definition must be equalities.

In the case $X = \mathbb{P}^N$ and $Y = H$ a hyperplane, Gathmann showed [Gat03b, Proposition 1.14] that $\overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d)$ is irreducible with dimension equal to the expected dimension:

$$\operatorname{vdim} \overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta) = \operatorname{vdim} \overline{\mathcal{M}}_{0,n}(X,\beta) - \sum_{i=1}^{n} \alpha_i$$

Hence it has a fundamental class from which one can define relative Gromov–Witten invariants. More generally if $Y \subseteq X$ is very ample one can use the embedding $X \hookrightarrow \mathbb{P}^N$ given by $|\mathcal{O}_X(Y)|$ to obtain a cartesian diagram:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta) & \longrightarrow & \overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d) \\ \downarrow & \square & \downarrow \\ \overline{\mathcal{M}}_{0,n}(X,\beta) & \xrightarrow{\varphi} & \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d) \end{array}$$

Then the fact that $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d)$ is smooth allows one to define a virtual class on $\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)$ by diagonal pull-back (see Appendix 2.7 of the current chapter):

$$[\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)]^{\text{virt}} := \varphi^! [\overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d)]$$

Thus one can define relative Gromov–Witten invariants in the usual way, by capping the virtual class with products of evaluation classes and psi classes.

In [Gat02] Gathmann establishes a recursion formula inside the Chow group of $\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)$. Since this formula forms the primary motivation for the work presented in this chapter, as well in Chapter 3, we will spend some time unpacking the key ideas. Full details and a more in-depth discussion can be found in [Gat02, §§2-4].

Consider therefore a smooth pair (X,Y) with Y very ample as above, and let α be a vector of tangency conditions with respect to Y. Choose a marking $x_k \in \{x_1,\ldots,x_n\}$ and let $e_k = (0,\ldots,1,\ldots,0)$. There is a closed embedding of virtual codimension one

$$\overline{\mathcal{M}}_{0,\alpha+e_k}(X|Y,\beta) \subseteq \overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)$$

which we think of as "increasing the tangency" at x_k by 1. Gathmann's recursion gives a formula for the class

$$[\overline{\mathcal{M}}_{0,\alpha+e_k}(X|Y,\beta)]^{\mathrm{virt}} \in \mathrm{A}_*\left(\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)\right)$$

in terms of boundary strata and tautological classes. To be more precise:

Theorem 2.2.12 ([Gat02, Theorem 2.6]). There is an equality

$$[\overline{\mathcal{M}}_{0,\alpha+e_k}(X|Y,\beta)]^{\mathrm{virt}} = \left(\alpha_k\psi_k + \mathrm{ev}_k^* Y\right) \cap [\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}} - [\mathcal{D}_{\alpha,k}(X|Y,\beta)]^{\mathrm{virt}}$$

in the Chow group of $\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)$.

The final term $\mathcal{D}_{\alpha,k}(X|Y,\beta)$ will be explained momentarily. The idea of the proof is as follows: one constructs a line bundle $\mathcal{L}$ on $\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)$, together with a global section s of $\mathcal{L}$ which gives, at each point of the moduli space, the α_kth derivative in the normal direction to Y of the stable map f, evaluated at the point x_k. Intuitively, the vanishing locus of s is precisely $\overline{\mathcal{M}}_{0,\alpha+e_k}(X|Y,\beta)$, whereas on the other hand one can compute directly that:

$$\mathrm{c}_1(\mathcal{L}) = \alpha_k\psi_k + \mathrm{ev}_k^* Y$$

This explains the first two terms in Theorem 2.2.12. What about the final term? This appears because the "intuition" given above for the vanishing locus of s is not quite correct: the vanishing locus actually consists of strictly more than $\overline{\mathcal{M}}_{0,\alpha+e_k}(X|Y,\beta)$. This is because it contains any stable map for which x_k belongs to an internal component (i.e. a component mapped inside Y), but not all such stable maps live inside $\overline{\mathcal{M}}_{0,\alpha+e_k}(X|Y,\beta)$.

Thus we find that the generic point of $\mathcal{D}_{\alpha,k}(X|Y,\beta)$ consists of stable maps in the complement

$$\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta) \setminus \overline{\mathcal{M}}_{0,\alpha+e_k}(X|Y,\beta)$$

and such that x_k belongs to an internal component of the source curve. Such a stable map $(C, x_1, \ldots, x_n, f)$ takes the form

$$C = C_0 \cup C_1 \cup \ldots \cup C_r$$

where C_0 is a subcurve containing x_k and mapping inside Y, and each C_i for $i \in \{1, \ldots, r\}$ is a subcurve which is not mapped inside Y, which intersects C_0 at a single node q_i, and which does not intersect any other C_j. Note that $r = 0$ is allowed, in which case the entire curve is mapped into Y.

Motivated by this picture, the locus $\mathcal{D}_{\alpha,k}(X|Y,\beta)$ is called the *comb locus* (C_0 being the handle of the comb and $C_1, \ldots, C_r$ being the teeth). It splits as a union of components (which we will also call comb loci), indexed by certain discrete data arising from the shape of the comb. To be more precise, the discrete data consists of quadruples (r, A, B, M) where:

- $r \geq 0$ is the number of teeth of the comb (from now on we will supress this in the notation);
- $A = (A_0, \ldots, A_r)$ is a partition of the set of markings

$$\{x_1, \ldots, x_n\} = A_0 \sqcup A_1 \sqcup \ldots \sqcup A_r$$

  such that $x_k \in A_0$;
- $B = (\beta_0, \ldots, \beta_r)$ is a splitting of the curve class

$$\beta = \beta_0 + \beta_1 + \ldots + \beta_r$$

  with $\beta_i > 0$ for $i \in \{1, \ldots, r\}$;
- $M = (m_1, \ldots, m_r)$ is a choice, for each $i \in \{1, \ldots, r\}$, of the tangency order $m_i > 0$ of C_i with Y at $q_i = C_0 \cap C_i$. Note that $m_i \leq Y \cdot \beta_i$.

The associated component of $\mathcal{D}_{\alpha,k}(X|Y, \beta)$ is then isomorphic to

$$\mathcal{D}(X|Y, A, B, M) := \overline{\mathcal{M}}_{0,A_0 \cup \{q_1,\ldots,q_r\}}(Y, \beta_0) \times_{Y^r} \prod_{i=1}^{r} \overline{\mathcal{M}}_{0,(\alpha|_{A_i}) \cup (m_i)}(X|Y, \beta_i)$$

where the fibre product is over the evaluation maps at the points $q_1, \ldots, q_r$. The notation $\alpha|_{A_i}$ indicates that we take the tangency vector $(\alpha_j : x_j \in A_i)$.

The discrete data (A, B, M) must be *stable*, in the sense that all the moduli spaces appearing in the above product are well-defined. Furthermore it must satisfy the following equation

$$Y \cdot \beta_0 + \sum_{i=1}^{r} m_i = \sum_{x_j \in A_0} \alpha_j$$

which ensures that

$$\mathcal{D}(X|Y, A, B, M) \subseteq \overline{\mathcal{M}}_{0,\alpha}(X|Y, \beta)$$

$$\mathcal{D}(X|Y, A, B, M) \nsubseteq \overline{\mathcal{M}}_{0,\alpha+e_k}(X|Y, \beta)$$

and that the first inclusion has virtual codimension 1. The virtual class on each comb locus is defined as a certain multiple of the class induced by the various factors

$$[\mathcal{D}(X|Y,A,B,M)]^{\mathrm{virt}} = \left(\frac{m_1\cdots m_r}{r!}\right)\Delta_r^!\left([\overline{\mathcal{M}}_{0,A_0\cup\{q_1,\ldots,q_r\}}(Y,\beta_0)]^{\mathrm{virt}}\times\prod_{i=1}^r[\overline{\mathcal{M}}^{\mathrm{Gat}}_{0,(\alpha|_{A_i})\cup(m_i)}(X|Y,\beta_i)]^{\mathrm{virt}}\right)$$

where $\Delta_r : Y^r \hookrightarrow Y^r \times Y^r$ is the diagonal embedding. The total comb locus is the union of these loci

$$\mathcal{D}_{\alpha,k}(X|Y,\beta) = \coprod_{(A,B,M)} \mathcal{D}(X|Y,A,B,M)$$

with virtual class inherited from the components:

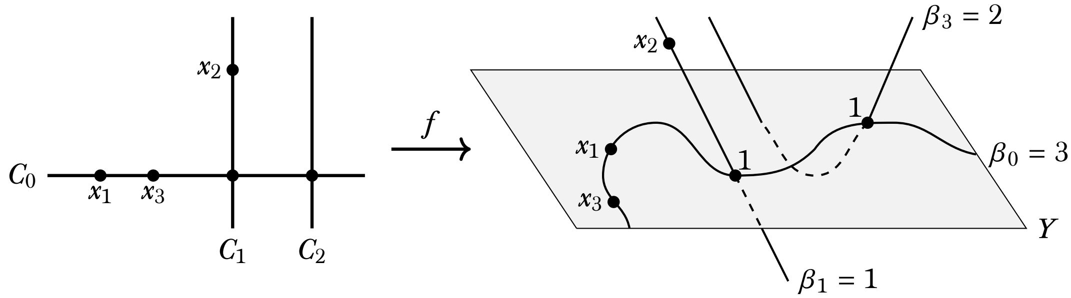
$$[\mathcal{D}_{\alpha,k}(X|Y,\beta)]^{\mathrm{virt}} = \sum_{(A,B,M)} [\mathcal{D}(X|Y,A,B,M)]^{\mathrm{virt}}$$

This is the class appearing in the statement of Theorem 2.2.12.

Example 2.2.13. Take $X = \mathbb{P}^3$ and $Y = H$ a hyperplane, and consider the following moduli space:

$$\overline{\mathcal{M}}_{0,(1,0,4)}(\mathbb{P}^3|H,6)$$

Take $x_k = x_3$, so we are looking for comb loci for which $x_k \in C_0$. One possible object of the comb locus is given by the following picture:



This gives a generic element of $\mathcal{D}(X|Y,A,B,M)$, where the discrete data A,B,M is given by:

- $r = 2$;
- $A_0 = \{x_1, x_3\}, A_1 = \{x_2\}, A_2 = \emptyset$;
- $\beta_0 = 3, \beta_1 = 1, \beta_2 = 2$;
- $m_1 = 1, m_2 = 1$.

Notice that the all-important equation

$$Y \cdot \beta_0 + \sum_{i=1}^{r} m_i = \sum_{x_j \in C_0} \alpha_j$$

is satisfied.

This completes the description of Gathmann's recursion. The important thing to notice here is the following: the definition of the virtual class on the comb locus means that any integral over it can be expressed in terms of the absolute invariants of Y and the relative invariants of (X, Y). Furthermore the relative invariants which appear in this way all have strictly lower tangency than the $\alpha + e_k$ appearing on the left-hand side of Theorem 2.2.12. Therefore one can apply this theorem recursively, until all of the tangencies have been reduced to zero. Since a relative invariant of tangency zero is the same thing as an absolute invariant, this provides an algorithm for expressing any relative invariant of (X, Y) in terms of the absolute invariants of X and of Y. The absolute invariants of Y which occur can then be expressed in terms of the absolute invariants of X via the quantum Lefschetz hyperplane principle. Thus, one obtains an algorithm for computing any relative invariant of (X, Y) in terms of the absolute invariants of X. For more details on this, see [Gat02, §5].

Remark 2.2.14. There are many other approaches to defining relative stable maps besides Gathmann's: the moduli space of maps to expanded degenerations of J. Li [Li01] [Li02], the twisted stable maps of D. Abramovich and B. Fantechi [AF16], the logarithmic stable maps with expansions of B. Kim [Kim10] and the logarithmic stable maps (without expansions) of M. Gross and B. Siebert [GS13], Q. Chen [Che14] and D. Abramovich and Q. Chen [AC14]. However, the invariants defined via these theories are all known to coincide [AMW14] [Gat03a], so the choice of which moduli space to work with mainly depends on one's intended application.

2.2.3. **Definition of relative stable quasimaps.** We now give the main definition of this chapter. From here on X will denote a smooth projective toric variety and $Y \subseteq X$ a very ample hypersurface. We *do not* require that Y is toric. Consider the line bundle $\mathcal{O}_X(Y)$ and the section s_Y cutting out Y. By [Cox95b] we have a natural isomorphism of $\mathbb{C}$-vector spaces

$$\mathrm{H}^0(X, \mathcal{O}_X(Y)) = \left\langle \prod_{\rho} z_\rho^{a_\rho} : \Sigma_\rho a_\rho [D_\rho] = [Y] \right\rangle_{\mathbb{C}}$$

where the z_ρ for $\rho \in \Sigma(1)$ are the generators of the Cox ring of X and the a_ρ are non-negative integers. We can therefore write s_Y as

$$s_Y = \sum_{\underline{a}=(a_\rho)} \lambda_{\underline{a}} \prod_\rho z_\rho^{a_\rho}$$

for some scalars $\lambda_{\underline{a}} \in \mathbb{C}$. The idea is that a quasimap

$$\big((C, x_1, \ldots, x_n), (L_\rho, u_\rho)_{\rho\in\Sigma(1)}, (\varphi_m)_{m\in M}\big)$$

should "map" a point $x \in C$ into Y if and only if the section

$$u_Y := \sum_{\underline{a}} \lambda_{\underline{a}} \prod_\rho u_\rho^{a_\rho} \tag{2.2.1}$$

vanishes at x. We now explain how to make sense of expression (2.2.1). For each $\underline{a}$ we have a well-defined section

$$u_{\underline{a}} := \lambda_{\underline{a}} \prod_\rho u_\rho^{a_\rho} \in \mathrm{H}^0(C, \otimes_\rho L_\rho^{\otimes a_\rho})$$

and if we have $\underline{a}$ and $\underline{b}$ such that $\sum_\rho a_\rho[D_\rho] = [Y] = \sum_\rho b_\rho[D_\rho]$ then these divisors differ by an element m of M. Thus the isomorphism φ_m allows us to view the sections $u_{\underline{a}}$ and $u_{\underline{b}}$ as sections of the same bundle, which we denote by L_Y. Then we can sum these together to obtain u_Y. There is a choice involved here, but up to isomorphism it does not matter; see the proof of functoriality in Appendix 2.6.1 for more details.

The upshot is that we obtain a line bundle L_Y on C, which plays the role of the "pull-back" of $\mathcal{O}_X(Y)$ along the "map" $C \to X$, and a global section

$$u_Y \in \mathrm{H}^0(C, L_Y)$$

which plays the role of the "pull-back" of s_Y.

Definition 2.2.15. With notation as above, let $n \geq 2$ be a number of marked points, $\beta \in \mathrm{H}_2^+(X)$ be an effective curve class and $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a collection of non-negative integers such that $\Sigma_i \alpha_i \leq Y \cdot \beta$. The *moduli space of relative stable quasimaps*

$$\mathcal{Q}_{0,\alpha}(X|Y, \beta) \subseteq \mathcal{Q}_{0,n}(X, \beta)$$

is defined to be the locus of quasimaps

$$\big((C \to S, (x_i : S \to C)_{i=1}^n), (L_\rho, u_\rho)_{\rho\in\Sigma(1)}, (\varphi_m)_{m\in M}\big)$$

such that:

(1) if x_i is a marking such that $\alpha_i > 0$, then $x_i^* u_Y = 0$;
(2) if we let $u_Y^*(0) \in \mathrm{A}^0(u_Y^{-1}(0) \to S)$ denote the class defined by the Gysin map for L_Y, then the difference $u_Y^*(0) - \Sigma_i \alpha_i [x_i]$ is an effective class.

The class $u_Y^*(0)$ is defined as follows. Consider the cartesian diagram

$$\begin{array}{ccc} u_Y^{-1}(0) & \longrightarrow & C \\ \downarrow & \square & \downarrow u_Y \\ C & \xrightarrow{0_Y} & L_Y \end{array}$$

where 0_Y is the zero section. There is a Gysin map [Ful98, §2.6]

$$0_Y^! : \mathrm{A}_*(C) \to \mathrm{A}_*(u_Y^{-1}(0))$$

and we define $u_Y^*(0) := 0_Y^!([C])$.

Remark 2.2.16. As in the case of relative stable maps (see §2.2.2) there is an equivalent definition which is more useful in practice: a quasimap is a relative quasimap if and only if for every connected component Z of $u_Y^{-1}(0)$ we have that:

(1) if Z is a point and is equal to a marked point x_i, then the order of vanishing of u_Y at x_i is greater than or equal to α_i;
(2) if Z is one-dimensional (hence a union of irreducible components) and if we let $C^{(i)}$ for $1 \leq i \leq r$ denote the irreducible components of C adjacent to Z and $m^{(i)}$ the order of vanishing of u_Y at the node $Z \cap C^{(i)}$, then:

$$\deg L_Y|_Z + \sum_{i=1}^{r} m^{(i)} \geq \sum_{x_i \in Z} \alpha_i \tag{**}$$

Remark 2.2.17. The above disussion also makes sense for ϵ-stable quasimaps where $\epsilon > 0$ is an arbitrary rational number. We therefore have a notion of *ϵ-stable relative quasimap.* For $\epsilon = 0+$ we recover relative quasimaps as above, whereas for $\epsilon > 1$ we recover relative stable maps in the sense of Gathmann.

For simplicity we restrict ourselves to the case $\epsilon = 0+$. However, all of the arguments can be adapted to the general case. As ϵ increases, the recursion formula (see §2.4) becomes progressively more complicated due to the presence of rational tails of lower and lower degree. Consequently the quantum Lefschetz theorem (see §2.5) also becomes more complicated.

2.3. Recursion formula for $\mathbb{P}^N$ relative a hyperplane

At this stage we do not know much about the moduli space of relative quasimaps. In this section we will examine the case $X = \mathbb{P}^N$ and $Y = H$ a hyperplane in detail.

2.3.1. **Basic properties of the moduli space.** We now show that the moduli space

$$\mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H, d)$$

is irreducible of the expected dimension, and thus admits a fundamental class. We then prove a recursion formula for these fundamental classes by pushing forward Gathmann's recursion formula along the comparison morphism:

$$\chi : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N, d) \to \mathcal{Q}_{0,n}(\mathbb{P}^N, d)$$

Let us briefly recall what this morphism does. Every stable map defines a quasimap which is stable except for the fact that it may have rational tails. χ has the effect of contracting these rational tails and introducing a basepoint at the corresponding node, with multiplicity equal to the degree of the rational tail.

For the remainder of this section we set $X = \mathbb{P}^N$, denote the projective co-ordinates on X by $[z_0 : \cdots : z_N]$ and set $Y = H = \{z_0 = 0\}$. Given a quasimap

$$\big((C, x_1, \ldots, x_n), L, u_0, \ldots, u_N\big) \in \mathcal{Q}_{0,n}(\mathbb{P}^N, d)$$

the line bundle L_Y of the previous section is equal to L and the section u_Y is equal to u_0.

Lemma 2.3.1. The comparison morphism restricts to a morphism

$$\chi : \overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H, d) \to \mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H, d)$$

Proof. We need to verify that a relative stable map is sent to a relative stable quasimap by χ. Since the contraction of a rational tail R always occurs away from the markings, we only need to examine the internal components Z of the quasimap. To be more precise, we have to show that the inequality (**) is satisfied, using the fact that the inequality (*) is satisfied by the stable map that we started with. Let us describe this stable map around Z. For each basepoint x on Z there is a rational tail R of the stable map attached to Z at x. This is either internal (mapped into H) or external (not mapped entirely into H).

If R is internal then both R and Z live inside the same connected component Z' of $f^{-1}(H)$. Applying χ has the effect of contracting R and increasing the degree of the line bundle on Z by

$H \cdot f_*[R]$. Thus the left hand side of inequality (*) is left unchanged, and since the right hand side is also unaltered we obtain inequality (**).

On the other hand if R is external then the multiplicity $m^{(R)}$ of $R \cap Z$ satisfies:

$$m^{(R)} \leq H \cdot f_*[R]$$

Since applying χ has the effect of replacing $m^{(R)}$ by $H \cdot f_*[R]$ in the left hand side of (*), inequality (**) holds a fortiori for the quasimap. Thus we obtain a morphism from the relative stable map space to the relative quasimap space, as claimed. □

Let us denote by

$$\mathcal{Q}^{\circ}_{0,\alpha}(\mathbb{P}^N|H,d) \subseteq \mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)$$

the *nice locus*, consisting of those quasimaps with irreducible source curve $C \cong \mathbb{P}^1$ and no basepoints (so that we have an actual map $u : C \to \mathbb{P}^N$) such that the curve is not mapped inside H and u has tangency at least α_i to H at the marking x_i.

This is an irreducible, locally closed substack of $\mathcal{Q}_{0,n}(\mathbb{P}^N,d)$ of codimension $\Sigma_i \alpha_i$, by essentially the same argument as in [Gat02, Lemma 1.8]. In fact it is isomorphic to the nice locus inside the stable map space, denoted $\mathcal{M}_{0,\alpha}(\mathbb{P}^N|H,d)$ by Gathmann (see [Gat02, Def. 1.6]); the stricter stability condition has no effect when the source curve is irreducible.

Lemma 2.3.2. $\mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)$ is equal to the closure of the nice locus $\mathcal{Q}^{\circ}_{0,\alpha}(\mathbb{P}^N|H,d)$ inside $\mathcal{Q}_{0,n}(\mathbb{P}^N,d)$.

Proof. $\mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d) \subseteq \overline{\mathcal{Q}^{\circ}_{0,\alpha}(\mathbb{P}^N|H,d)}$: we show that any relative stable quasimap can be infinitesimally deformed to a relative stable quasimap with no basepoints. This is in particular a relative stable map; we then appeal to [Gat02, Prop. 1.14] to deform this stable map and obtain a point in the nice locus. Since this deformation does not introduce any rational tails, this is also a deformation of quasimaps, and the statement follows.

We induct on the number of components containing at least one base-point. Suppose this number is non-zero (otherwise there is nothing to prove) and pick such a component C_0, with base-points $y_1,\ldots,y_k$. Recall that this means that $u_i(y_j) = 0$ for all i and j. We will deform the section $u_N|_{C_0}$ to a new section $u'_N|_{C_0}$ in such a way that $u'_N|_{C_0}(y_j) \neq 0$ and in such a way that we do not introduce any new basepoints. Notice that since the relative condition only depends on u_0, the resulting deformed quasimap will still be a relative quasimap.

Now, by nondegeneracy and the fact that there exists a basepoint, we must have $\deg(L|_{C_0}) > 0$, and since $C_0 \cong \mathbb{P}^1$ we can find a section w_0 of $L|_{C_0} \cong \mathcal{O}_{\mathbb{P}^1}(d_0)$ not vanishing at any of the basepoints y_i. We then set

$$u'_N|_{C_0} := u_N|_{C_0} + \epsilon w_0$$

and $u'_i|_{C_0} = u_i|_{C_0}$ for all other i. Notice that $u'_N|_{C_0}(y_j) \neq 0$ for all j as claimed. It is also clear that we do not introduce any new basepoints, since $u'_N|_{C_0}(y) = 0$ implies $u_N|_{C_0}(y) = 0$ (put differently: being a basepoint is a closed condition).

It remains to extend the section $u'_N|_{C_0}$ to a section u'_N on the whole curve. Let $C_1, \ldots, C_r$ be the components of C adjacent to C_0 and let $q_i = C_0 \cap C_i$. We need to modify the sections $u_N|_{C_i}$ in such a way that $u'_N|_{C_i}(q_i) = u'_N|_{C_0}(q_i)$.

By nondegeneracy, we can choose a section w_i of $L|_{C_i}$ such that $w_i(q_i) \neq 0$. Then set:

$$u'_N|_{C_i} := u_N|_{C_i} + \epsilon \left(\frac{w_0(q_i)}{w_i(q_i)} \right) \cdot w_i$$

Then indeed we have:

$$u'_N|_{C_i}(q_i) = u_N(q_i) + \epsilon \left(\frac{w_0(q_i)}{w_i(q_i)} \right) \cdot w_i(q_i) = u_N(q_i) + \epsilon w_0(q_i) = u'_N|_{C_0}(q_i)$$

We can continue this process, replacing C_0 by C_i; since the genus of the curve is zero there are no cycles in the dual intersection graph, and so we will never come to the same component twice. In this way we obtain a new quasimap

$$\left((C, x_1, \ldots, x_n), L, u_0, \ldots, u_{N-1}, u'_N\right)$$

over $\operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$ which has no basepoints on C_0. We can repeat this process for all the components of C (using higher powers of ϵ each time in order to ensure that we never introduce additional basepoints) and thus we obtain an infinitesimal deformation of our original quasimap which has no basepoints, as required.

$\overline{\mathcal{Q}^{\circ}_{0,\alpha}(\mathbb{P}^N|H,d)} \subseteq \mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)$: consider a family of stable quasimaps over a smooth curve S, such that the generic fibre lies in the nice locus. We may blow-up the source curve (a fibered surface over S) in the locus of basepoints (which consists of finitely many smooth points of the central fiber) and repeat this process a finite number of times in order to obtain an actual morphism to $\mathbb{P}^N$. This has the effect of adding rational tails at the basepoints in the central fibre. If the morphism is constant on any of these rational tails we may contract them, and thus we obtain a family of stable maps which pushes down along χ to our original family of quasimaps.

The general fibre is not modified at all, and so is still in the nice locus. By [Gat02, Lemma 1.9] it follows that the central fibre is a relative stable map, and then by applying χ and appealing to Lemma 2.3.1 it follows that the same is true for the central fibre of the family of quasimaps. ◻

Corollary 2.3.3. The moduli space $\mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)$ is irreducible of the expected dimension. Hence it has a fundamental class.

Proof. This holds because the moduli space is equal to the closure of the nice locus, which is irreducible of the expected dimension. ◻

Since the moduli space of relative quasimaps has a fundamental class, we can define *relative quasimap invariants* for the pair $(\mathbb{P}^N, H)$:

$$\left\langle \gamma_1\psi_1^{k_1},\ldots,\gamma_n\psi_n^{k_n}\right\rangle_{0,\alpha,d}^{\mathbb{P}^N|H} := \int_{[\mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)]}\prod_{i=1}^{n}\mathrm{ev}_i^*\gamma_i\cdot\psi_i^{k_i}$$

We will now establish a number of properties of the fundamental class.

Corollary 2.3.4. The comparison morphism from relative stable maps to relative quasimaps is birational. In particular it sends the fundamental class to the fundamental class, and thus the invariants coincide.

Proof. This follows because the comparison morphism restricts to an isomorphism on the nice locus, which by Lemma 2.3.2 is a dense open subset of both spaces. ◻

2.3.2. **Proof of the recursion formula.** We wish to obtain a recursion formula relating the quasimap invariants of multiplicity α with the quasimap invariants of multiplicity $\alpha+e_k$, as in [Gat02, Theorem 2.6]. For $m=\alpha_k+1$ the following section (of the pull-back of the jet bundle of the universal line bundle)

$$\sigma_k^m := x_k^* d_{\mathcal{C}/\mathcal{Q}}^m(u_0) \in \mathrm{H}^0(\mathcal{Q}, x_k^*\mathcal{P}_{\mathcal{C}/\overline{\mathcal{Q}}}^m(\mathcal{L}))$$

vanishes along $\mathcal{Q}_{0,\alpha+e_k}(\mathbb{P}^N|H,d)$ inside $\mathcal{Q}=\mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)$, and also along a number of *comb loci.* The latter parametrise quasimaps for which x_k belongs to an internal component $Z\subseteq C$ (a connected component of the vanishing locus of u_0), such that:

$$\deg(L|_Z)+\sum_{i=1}^{r}m^{(i)}=\sum_{x_i\in Z}\alpha_i$$

The strong stability condition means that quasimaps in the comb loci cannot contain any rational tails; this is really the only difference with the case of stable maps.

Indeed, we can push forward Gathmann's recursion formula for stable maps along the comparison morphism

$$\chi\colon \overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d) \to \mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)$$

and, due to Corollary 2.3.4 above, the only terms which change are the comb loci containing rational tails. In fact these disappear, since the restriction of the comparison map to these loci has positive-dimensional fibres:

Lemma 2.3.5. Consider a rational tail component in the comb locus of the moduli space of stable maps, i.e. a moduli space of the form:

$$\overline{\mathcal{M}}_{0,(m^{(i)})}(\mathbb{P}^N|H,d)$$

and assume that $Nd > 1$. Then

$$\dim\left([\overline{\mathcal{M}}_{0,(m^{(i)})}(\mathbb{P}^N|H,d)] \cap \mathrm{ev}_1^*(\mathrm{pt}_H)\right) > 0$$

where $\mathrm{pt}_H \in \mathrm{A}^{N-1}(H)$ is a point class. Thus the pushforward along χ of any comb locus with a rational tail is zero.

Proof. This is a simple dimension count. We have

$$\begin{aligned}\dim\left([\overline{\mathcal{M}}_{0,(m^{(i)})}(\mathbb{P}^N|H,d)] \cap \mathrm{ev}_1^*(\mathrm{pt}_H)\right) &= (N-3) + d(N+1) + (1-m^{(i)}) - (N-1) \\ &= (Nd-1) + (d-m^{(i)})\end{aligned}$$

from which the lemma follows because $m^{(i)} \leq d$. □

Remark 2.3.6. With an eye to the future, we remark that these rational tail components contribute nontrivially to the Gromov–Witten invariants of a Calabi–Yau hypersurface in projective space, and so their absence from the quasimap recursion formula accounts for the divergence between Gromov–Witten and quasimap invariants in the Calabi–Yau case [Gat03b, Rmk. 1.6].

Since we wish to apply the projection formula to Gathmann's recursion relation, we should express the cohomological terms which appear as pull-backs:

Lemma 2.3.7. We have:

$$\chi^*(\psi_k) = \psi_k$$

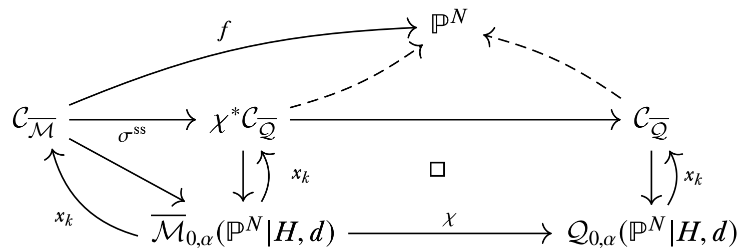
$$\chi^*(\mathrm{ev}_k^* H) = \mathrm{ev}_k^* H$$

Proof. It suffices to show that:

$$\chi^* x_k^* \omega_{\mathcal{C}/\mathcal{Q}} = x_k^* \omega_{\mathcal{C}/\mathcal{M}}$$

$$\chi^* x_k^* \mathcal{L} = \mathrm{ev}_k^* \mathcal{O}_{\mathbb{P}^N}(H)$$

This follows by considering the following diagram:



where σ^{ss} is the strong stabilisation map which contracts the rational tails. Note that σ^{ss} is an isomorphism near the markings. □

Proposition 2.3.8. Define the *quasimap comb locus* $\mathcal{D}^{\mathcal{Q}}_{\alpha,k}(\mathbb{P}^N|H,d)$ as the union of the moduli spaces

$$\mathcal{D}^{\mathcal{Q}}(\mathbb{P}^N|H,A,B,M) := \mathcal{Q}_{0,A^{(0)}\cup\{q_1^0,\ldots,q_r^0\}}(H,d_0) \times_{H^r} \prod_{i=1}^{r} \mathcal{Q}_{0,\alpha^{(i)}\cup(m^{(i)})}(\mathbb{P}^N|H,d_i)$$

where the union runs over all splittings $A = (A^{(0)},\ldots,A^{(r)})$ of the markings (inducing a splitting $(\alpha^{(0)},\ldots,\alpha^{(r)})$ of the corresponding tangency conditions), $B = (d_0,\ldots,d_r)$ of the degree and all valid multiplicites $M = (m^{(1)},\ldots,m^{(r)})$ such that the above spaces are all well-defined (in particular we require that $|A^{(0)}| + r$ and $|A^{(i)}| + 1$ are all ≥ 2) and such that

$$d_0 + \sum_{i=1}^{r} m^{(i)} = \sum \alpha^{(0)}$$

Write $[\mathcal{D}^{\mathcal{Q}}_{\alpha,k}(\mathbb{P}^N|H,d)]$ for the sum of the (product) fundamental classes, where each term is weighted by:

$$\frac{m^{(1)}\cdots m^{(r)}}{r!}$$

Then

$$(\alpha_k\psi_k + \mathrm{ev}_k^* H)\cdot[\mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)] = [\mathcal{Q}_{0,\alpha+e_k}(\mathbb{P}^N|H,d)] + [\mathcal{D}^{\mathcal{Q}}_{\alpha,k}(\mathbb{P}^N|H,d)].$$

Proof. This follows from [Gat02, Thm. 2.6] by pushing forward along χ, using the projection fomula and Lemmas 2.3.4, 2.3.5 and 2.3.7 . □

Remark 2.3.9. In the discussion above we have implicitly used the fact that there exists a commuting diagram of comb loci:

$$\begin{array}{ccc} \mathcal{D}^{\mathcal{M}}(\mathbb{P}^N|H,A,B,M) & \longrightarrow & \overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d) \\ \downarrow & & \downarrow \\ \mathcal{D}^{\mathcal{Q}}(\mathbb{P}^N|H,A,B,M) & \longrightarrow & \mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d) \end{array}$$

The vertical arrow on the left is a product of comparison morphisms (notice that $H \simeq \mathbb{P}^{r-1}$). The horizontal arrow at the top is the glueing morphism which glues together the various pieces of the comb to produce a single relative stable map. Here we explain how to define the corresponding glueing morphism for quasimaps, that is, the bottom horizontal arrow.

Suppose for simplicity that $r = 1$ and consider an element of the quasimap comb locus, consisting of two quasimaps:

$$\big((C^0, x_1^0, \ldots, x_{n_0}^0, q^0), L^0, u_0^0, \ldots, u_N^0\big)$$
$$\big((C^1, x_1^1, \ldots, x_{n_1}^1, q^1), L^1, u_0^1, \ldots, u_N^1\big)$$

such that $u^0(q^0) = u^1(q^1)$ in $\mathbb{P}^N$. We want to glue these quasimaps together at q^0, q^1. The definition of the curve is obvious; we simply take:

$$C = C^0 \,{}_{q^0}\!\sqcup_{q^1} C^1$$

On the other hand, glueing together the line bundles L^0 and L^1 to obtain a line bundle L over C requires a choice of scalar $\lambda \in \mathbb{G}_{\mathrm{m}}$, in order to match up the fibres over q^i. Furthermore if the sections are to extend as well then this scalar must be chosen in such a way that it takes $(u_0^0(q^0), \ldots, u_N^0(q^0)) \in (L^0_{q^0})^{\oplus(N+1)}$ to $(u_0^1(q^1), \ldots, u_N^1(q^1)) \in (L^1_{q^1})^{\oplus(N+1)}$. Since neither q^0 nor q^1 are basepoints (because they are markings), these tuples are nonzero, and so λ is unique if it exists. Furthermore it exists if and only if these tuples belong to the same $\mathbb{G}_{\mathrm{m}}$-orbit in $\mathbb{A}^{N+1}$. This is precisely the statement that $u^0(q^0) = u^1(q^1) \in \mathbb{P}^N$.

Similar arguments apply for $r > 1$, and for more general toric varieties.

2.4. Recursion formula in the general case

In this section we prove the main result of this chapter: a recursion formula for relative quasimap invariants of a general pair (X, Y).

Theorem 2.4.1. Let X be a smooth and proper toric variety and let $Y \subseteq X$ be a very ample hypersurface (not necessarily toric). Then

$$(\alpha_k\psi_k + \mathrm{ev}_k^*[Y]) \cap [\mathcal{Q}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}} = [\mathcal{Q}_{0,\alpha+e_k}(X|Y,\beta)]^{\mathrm{virt}} + [\mathcal{D}^{\mathcal{Q}}_{\alpha,k}(X|Y,\beta)]^{\mathrm{virt}}$$

in the Chow group of $\mathcal{Q}_{0,\alpha}(X|Y,\beta)$.

We begin by defining the terms that appear in the statement.

2.4.1. **The virtual class on the space of relative quasimaps**. Let X and Y be as in the statement of Theorem 2.4.1. The complete linear system associated to $\mathcal{O}_X(Y)$ defines an embedding $i : X \hookrightarrow \mathbb{P}^N$ such that $i^{-1}(H) = Y$ for a certain hyperplane H. By the functoriality property of quasimap spaces (see Appendix 2.6.1) we have a map:

$$k := \mathcal{Q}(i) : \mathcal{Q}_{0,n}(X,\beta) \to \mathcal{Q}_{0,n}(\mathbb{P}^N, d)$$

where $d = i_*\beta$. Because $\mathcal{Q}_{0,n}(\mathbb{P}^N, d)$ is smooth, k admits a compatible perfect obstruction theory (see Appendix 2.6.2), so we have a notion of virtual pull-back along k.

Remark 2.4.1. I. Ciocan-Fontanine has kindly pointed out that, contrary to the case of stable maps, k might not be a closed embedding, even though i is. Consider the Segre embedding:

$$\mathbb{P}^1 \times \mathbb{P}^1 \overset{i}{\hookrightarrow} \mathbb{P}^3$$

$$([x:y],[z:w]) \mapsto [xz : xw : yz : yw]$$

Consider the induced morphism between quasimap spaces

$$k\colon \mathcal{Q}_{0,3}(\mathbb{P}^1 \times \mathbb{P}^1, (2,2)) \to \mathcal{Q}_{0,3}(\mathbb{P}^3, 4)$$

and the following two objects of $\mathcal{Q}_{0,3}(\mathbb{P}^1 \times \mathbb{P}^1, (2,2))$:

$$\left(\left(\mathbb{P}^1_{[s:t]}, 0, 1, \infty\right), \left(L_1 = \mathcal{O}_{\mathbb{P}^1}(2), u_1 = s^2, v_1 = st\right), \left(L_2 = \mathcal{O}_{\mathbb{P}^1}(2), u_2 = st, v_2 = t^2\right)\right)$$
$$\left(\left(\mathbb{P}^1_{[s:t]}, 0, 1, \infty\right), \left(L_1 = \mathcal{O}_{\mathbb{P}^1}(2), u_1 = st, v_1 = t^2\right), \left(L_2 = \mathcal{O}_{\mathbb{P}^1}(2), u_2 = s^2, v_2 = st\right)\right)$$

These two quasimaps are non-isomorphic, but they both map to the same object under k, namely:

$$\left(\left(\mathbb{P}^1_{[s:t]}, 0, 1, \infty\right), \left(L = \mathcal{O}_{\mathbb{P}^1}(4), z_0 = s^3t, z_1 = s^2t^2, z_2 = s^2t^2, z_3 = st^3\right)\right)$$

Notice that this only happens on the locus of quasimaps with basepoints.

It is easy to show that k restricts to a morphism between moduli space of relative quasimaps, and thus we have a diagram

$$\begin{array}{ccc} \mathcal{Q}_{0,\alpha}(X|Y,\beta) & \xrightarrow{g} & \mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d) \\ \Big\downarrow f & \square & \Big\downarrow j \\ \mathcal{Q}_{0,n}(X,\beta) & \xrightarrow{k} & \mathcal{Q}_{0,n}(\mathbb{P}^N,d) \end{array}$$

which one can show is cartesian. As such we can define a virtual class on $\mathcal{Q}_{0,\alpha}(X|Y,\beta)$ by pullback along k:

$$[\mathcal{Q}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}} := k^![\mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)]$$

We use this class to define relative quasimap invariants in general:

$$\langle \gamma_1\psi_1^{k_1},\ldots,\gamma_n\psi_n^{k_n}\rangle_{0,\alpha,\beta}^{X|Y} := \int_{[\mathcal{Q}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i)\cdot\psi_i^{k_i}$$

These invariants will play a role in our proof of the quasimap Lefschetz formula in §2.5.

2.4.2. **Relative spaces pull back**. The idea is to prove the recursion formula for general (X,Y) by pulling back the formula for $(\mathbb{P}^N,H)$ along k. In order to do this, we need to understand how the various virtual classes involved in the formula pull back along this map. The first two terms pull back by the very definition of the virtual class:

Lemma 2.4.2. $k^![\mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)] = [\mathcal{Q}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}}$

It thus remains to consider the third term, namely the virtual class of the comb locus. This is the technical heart of the proof.

2.4.3. **Comb loci pull back**. As in the previous section, we define $\mathcal{D}^{\mathcal{Q}}_{\alpha,k}(X|Y,\beta)$ to be the union of the moduli spaces

$$\mathcal{D}^{\mathcal{Q}}(X|Y,A,B,M) := \mathcal{Q}_{0,A^{(0)}\cup\{q_1,\ldots,q_r\}}(Y,\beta^{(0)}) \times_{Y^r} \prod_{i=1}^{r} \mathcal{Q}_{0,\alpha^{(i)}\cup(m_i)}(X|Y,\beta^{(i)})$$

where the union runs over all splittings $A=(A^{(0)},\ldots,A^{(r)})$ of the markings (inducing a splitting $(\alpha^{(0)},\ldots,\alpha^{(r)})$ of the corresponding tangency requirements), $B=(\beta^{(0)},\ldots,\beta^{(r)})$ of the curve class β and all valid multiplicities $M=(m^{(1)},\ldots,m^{(r)})$ such that the above spaces are non-empty and such that:

$$Y\cdot\beta^{(0)}+\sum_{i=1}^{r} m^{(i)} = \sum \alpha^{(0)}$$

We refer to the $\mathcal{D}^{\mathcal{Q}}(X|Y,A,B,M)$ as *comb loci.*

Remark 2.4.3. Note that Y is not in general toric, and so we should clarify what we mean by:

$$\mathcal{Q}(Y) = \mathcal{Q}_{0,A^{(0)}\cup\{q_1,\ldots,q_n\}}(Y,\beta^{(0)})$$

There are two possibilities here: one is to *define* this space as the cartesian product

$$\begin{array}{ccc} \mathcal{Q}(Y) & \longrightarrow & \mathcal{Q}(H) \\ \downarrow & \square & \downarrow \\ \mathcal{Q}(X) & \xrightarrow{k} & \mathcal{Q}(\mathbb{P}^N) \end{array}$$

and equip it with the virtual class pulled back along k:

$$[\mathcal{Q}(Y)]^{\mathrm{virt}} := k^![\mathcal{Q}(H)]$$

Using this definition, $\mathcal{Q}(Y)$ consists of those quasimaps in $\mathcal{Q}(X)$ for which $u_Y \equiv 0$. This has obvious advantages from the point of view of our computations, but is conceptually unsatisfying.

On the other hand, X is a GIT quotient $\mathbb{A}^{\Sigma_X(1)} /\!\!/ \mathbb{G}_{\mathrm{m}}^r$, and $Y \subseteq X$ defines a $\mathbb{G}_{\mathrm{m}}^r$-invariant subvariety $C(Y)$ of $\mathbb{A}^{\Sigma_X(1)}$, which we call the *cone over Y*. Then Y is equal to the GIT quotient

$$Y = C(Y) /\!\!/ \mathbb{G}_{\mathrm{m}}^r$$

and so we may use the more general theory of quasimaps to GIT quotients [CFKM14] to define $\mathcal{Q}(Y)$ and its virtual class.

In fact these two definitions of $\mathcal{Q}(Y)$ agree: there exists an isomorphism between these moduli spaces which preserves the virtual classes. We show this in Appendix 2.6.4.

We now construct a virtual class on the comb locus $\mathcal{D}^{\mathcal{Q}}(X|Y,A,B,M)$. Consider the product (*not* the fibre product over Y^r)

$$\mathcal{E}^{\mathcal{Q}}(X|Y,A,B,M) := \mathcal{Q}_{0,A^{(0)}\cup\{q_1,\ldots,q_r\}}(Y,\beta^{(0)}) \times \prod_{i=1}^{r} \mathcal{Q}_{0,\alpha^{(i)}\cup(m_i)}(X|Y,\beta^{(i)})$$

which we may endow with the product virtual class (with weighting as before):

$$[\mathcal{E}^{\mathcal{Q}}(X|Y,A,B,M)]^{\mathrm{virt}} := \left(\frac{m^{(1)}\cdots m^{(r)}}{r!}\right)\cdot\left([\mathcal{Q}_{0,A^{(0)}\cup\{q_1,\ldots,q_r\}}(Y,\beta^{(0)})]^{\mathrm{virt}} \times \prod_{i=1}^{r}[\mathcal{Q}_{0,\alpha^{(i)}\cup(m_i)}(X|Y,\beta^{(i)})]^{\mathrm{virt}}\right)$$

We have the following cartesian diagram

$$\begin{array}{ccc} \mathcal{D}^{\mathcal{Q}}(X|Y,A,B,M) & \longrightarrow & \mathcal{E}^{\mathcal{Q}}(X|Y,A,B,M) \\ \downarrow & \square & \downarrow \\ X^r & \xrightarrow{\Delta_{X^r}} & X^r \times X^r \end{array}$$

and we can use this to define a virtual class on the comb locus:

$$[\mathcal{D}^{\mathcal{Q}}(X|Y,A,B,M)]^{\mathrm{virt}} := \Delta^{!}_{X^r}[\mathcal{E}^{\mathcal{Q}}(X|Y,A,B,M)]^{\mathrm{virt}}$$

The virtual class on the union $\mathcal{D}^{\mathcal{Q}}_{\alpha,k}(X|Y,\beta)$ of the comb loci is defined to be the sum of the virtual classes $[\mathcal{D}^{\mathcal{Q}}(X|Y,A,B,M)]^{\mathrm{virt}}$.

Remark 2.4.4. This is the same definition of the virtual class of the comb locus that we gave in §2.3.2 in the case $(X,Y) = (\mathbb{P}^N, H)$.

On the other hand, there is another cartesian diagram:

$$\begin{array}{ccc} \coprod_{B:\, i_*B=B'} \mathcal{D}^{\mathcal{Q}}(X|Y,A,B,M) & \longrightarrow & \mathcal{D}^{\mathcal{Q}}(\mathbb{P}^N|H,A,B',M) \\ \downarrow & \square & \downarrow \\ \mathcal{Q}_{0,n}(X,\beta) & \xrightarrow{\quad k \quad} & \mathcal{Q}_{0,n}(\mathbb{P}^N,d) \end{array}$$

Recall that we are trying to show that the virtual class of the comb locus pulls back nicely along k. The result that we need is:

Lemma 2.4.5. $k^![\mathcal{D}^{\mathcal{Q}}(\mathbb{P}^N|H,A,B',M)]^{\mathrm{virt}} = \sum_{B:i_*B=B'} [\mathcal{D}^{\mathcal{Q}}(X|Y,A,B,M)]^{\mathrm{virt}}$

For the proof of Lemma 2.4.5, let us introduce the following shorthand notation. We fix the the data of A, B', M and set:

$$\begin{array}{ll} \mathcal{D}(X|Y) := \coprod_{B:\, i_*B=B'} \mathcal{D}^{\mathcal{Q}}(X|Y,A,B,M) & \mathcal{D}(\mathbb{P}^N|H) := \mathcal{D}^{\mathcal{Q}}(\mathbb{P}^N|H,A,B',M) \\ \mathcal{E}(X|Y) := \coprod_{B:\, i_*B=B'} \mathcal{E}^{\mathcal{Q}}(X|Y,A,B,M) & \mathcal{E}(\mathbb{P}^N|H) := \mathcal{E}^{\mathcal{Q}}(\mathbb{P}^N|H,A,B',M) \\ \mathcal{D}(X) := \coprod_{B:\, i_*B=B'} \mathcal{D}^{\mathcal{Q}}(X,A,B) & \mathcal{D}(\mathbb{P}^N) := \mathcal{D}^{\mathcal{Q}}(\mathbb{P}^N,A,B') \\ \mathcal{E}(X) := \coprod_{B:\, i_*B=B'} \mathcal{E}^{\mathcal{Q}}(X,A,B) & \mathcal{E}(\mathbb{P}^N) := \mathcal{E}^{\mathcal{Q}}(\mathbb{P}^N,A,B') \\ \mathcal{Q}(X) := \mathcal{Q}_{0,n}(X,\beta) & \mathcal{Q}(\mathbb{P}^N) := \mathcal{Q}_{0,n}(\mathbb{P}^N,i_*\beta) \end{array}$$

Here $\mathcal{D}(X)$ and $\mathcal{E}(X)$ are the centipede loci introduced in Appendix 2.6.3; they are defined in the same way as the comb loci, except that we replace both the quasimaps to Y and the relative quasimaps to (X,Y) by quasimaps to X. There is a cartesian diagram

$$\begin{array}{ccc} \mathcal{E}(X|Y) & \longrightarrow & \mathcal{E}(\mathbb{P}^N|H) \\ \downarrow & \square & \downarrow \theta \\ \mathcal{E}(X) & \longrightarrow & \mathcal{E}(\mathbb{P}^N) \end{array}$$

and, since $\mathcal{E}(\mathbb{P}^N)$ is smooth and there is a natural fundamental class on $\mathcal{E}(\mathbb{P}^N|H)$, we have a diagonal pull-back morphism $\theta^! = \theta^!_\Delta$ (see Appendix 2.7).

Lemma 2.4.6. $[\mathcal{E}(X|Y)]^{\text{virt}} = \theta^![\mathcal{E}(X)]^{\text{virt}}$

Proof. It suffices to check that in the following cartesian diagram

$$\begin{array}{ccc} \mathcal{Q}(Y) & \longrightarrow & \mathcal{Q}(H) \\ \downarrow & \square & \downarrow \theta \\ \mathcal{Q}(X) & \longrightarrow & \mathcal{Q}(\mathbb{P}^N) \end{array}$$

we have $\theta^![\mathcal{Q}(X)]^{\text{virt}} = [\mathcal{Q}(Y)]^{\text{virt}}$. Depending on one's definition of $\mathcal{Q}(Y)$ (see Remark 2.4.3 above) this is either true by definition or is proved in Appendix 2.6.4. ◻

Now consider the following cartesian diagram

$$\begin{array}{ccccc} \mathcal{D}(X) & \longrightarrow & \mathcal{D}(\mathbb{P}^N) & \longrightarrow & \mathfrak{M}^{\text{wt}}_{A,B} \\ \downarrow \varphi_X & \square & \downarrow \varphi_{\mathbb{P}^N} & \square & \downarrow \psi \\ \mathcal{Q}(X) & \xrightarrow{k} & \mathcal{Q}(\mathbb{P}^N) & \longrightarrow & \mathfrak{M}^{\text{wt}}_{0,n,\beta} \end{array}$$

where $\mathfrak{M}^{\text{wt}}_{0,n,\beta}$ is the moduli space of prestable curves weighted by the class β [Cos06, §2] and:

$$\mathfrak{M}^{\text{wt}}_{A,B} := \mathfrak{M}^{\text{wt}}_{0,A^{(0)}\cup\{q^0_1,\ldots,q^0_r\},\beta^{(0)}} \times \prod_{i=1}^{r} \mathfrak{M}^{\text{wt}}_{0,A^{(i)}\cup\{q^1_i\},\beta^{(i)}}$$

The maps $\mathcal{D}(X) \to \mathfrak{M}^{\text{wt}}_{A,B}$ and $\mathcal{Q}(X) \to \mathfrak{M}^{\text{wt}}_{0,n,\beta}$ admit relative perfect obstruction theories which are the same as the usual perfect obstruction theories relative to the moduli spaces of *unweighted* curves. Furthermore the morphism ψ admits a perfect obstruction theory; see Appendix 2.6.3 for details. Thus there are virtual pull-back morphisms $\psi^!$, and by the splitting axiom (see Lemma 2.6.5) we have

$$[\mathcal{D}(X)]^{\text{virt}} := \Delta^!_{X^r}[\mathcal{E}(X)]^{\text{virt}} = \psi^![\mathcal{Q}(X)]^{\text{virt}}$$

Commutativity of virtual pull-backs then implies that:

$$[\mathcal{D}(X)]^{\text{virt}} = \psi^![\mathcal{Q}(X)]^{\text{virt}} = \psi^!k^![\mathcal{Q}(\mathbb{P}^N)] = k^!\psi^![\mathcal{Q}(\mathbb{P}^N)] = k^![\mathcal{D}(\mathbb{P}^N)] \tag{2.4.1}$$

Proof of Lemma 2.4.5. Putting all the preceding results together, we consider the cartesian diagram:

$$\begin{array}{ccccc}
\mathcal{D}(X|Y) & \longrightarrow & \mathcal{E}(X|Y) & \longrightarrow & \mathcal{E}(\mathbb{P}^N|H) \\
\downarrow & \square & \downarrow & \square & \downarrow \theta \\
\mathcal{D}(X) & \longrightarrow & \mathcal{E}(X) & \longrightarrow & \mathcal{E}(\mathbb{P}^N) \\
\downarrow & \square & \downarrow & & \\
X^r & \xrightarrow{\Delta_{X^r}} & X^r \times X^r & &
\end{array}$$

We then have:

$$\begin{aligned}
[\mathcal{D}(X|Y)]^{\mathrm{virt}} &= \Delta^!_{X^r}[\mathcal{E}(X|Y)]^{\mathrm{virt}} && \text{by definition} \\
&= \Delta^!_{X^r}\theta^![\mathcal{E}(X)]^{\mathrm{virt}} && \text{by Lemma 2.4.6} \\
&= \theta^!\Delta^!_{X^r}[\mathcal{E}(X)]^{\mathrm{virt}} && \text{by commutativity} \\
&= \theta^![\mathcal{D}(X)]^{\mathrm{virt}} && \text{by definition} \\
&= \theta^! k^![\mathcal{D}(\mathbb{P}^N)] && \text{by formula (2.4.1) above} \\
&= \theta^! k^! \Delta^!_{(\mathbb{P}^N)^r}[\mathcal{E}(\mathbb{P}^N)] && \text{by definition} \\
&= k^! \Delta^!_{(\mathbb{P}^N)^r}\theta^![\mathcal{E}(\mathbb{P}^N)] && \text{by commutativity} \\
&= k^! \Delta^!_{\mathbb{P}^{N^r}}[\mathcal{E}(\mathbb{P}^N|H)] && \text{by Lemma 2.4.6} \\
&= k^![\mathcal{D}(\mathbb{P}^N|H)] && \text{by definition}
\end{aligned}$$

Summing over all the components of $\mathcal{D}^{Q}_{\alpha,k}(\mathbb{P}^N|H,d)$ we obtain the result. □

Proof of Theorem 2.4.1. Apply $k^!$ to Proposition 2.3.8, using Lemmas 2.4.2 and 2.4.5. □

2.5. Quasimap quantum Lefschetz

The recursion formula shows that the relative quasimap invariants of (X,Y) are completely determined, in an algorithmic way, from the absolute invariants of X and Y; by repeatedly applying the recursion formula, we can remove all the tangency conditions, leaving us with an expression which only involves the invariants of X and Y.

However, we can do much more than this. In this section we will prove (two variations of) a *quantum Lefschetz theorem for quasimap invariants*, that is, a result which expresses the quasimap invariants of Y in terms of those of X. This is the quasimap analogue of the quantum Lefschetz hyperplane principle in Gromov–Witten theory and, on the face of it, has nothing to do with relative invariants.

2.5.1. **General quasimap Lefschetz theorem.** First we state the most general form of the theorem, without any additional assumptions on X and Y.

Theorem 2.5.1 (Quasimap quantum Lefschetz). Let X be a smooth projective toric variety and $Y \subseteq X$ a smooth very ample hypersurface. Then there is an explicit algorithm to recover the (restricted) absolute quasimap invariants of Y, as well as the relative quasimap invariants of (X, Y), from the absolute quasimap invariants of X.

The corresponding result in Gromov–Witten theory is due to Gathmann [Gat03a, Corollary 2.5.6]; the proof we present in the quasimap setting is very similar to his. The term *"restricted"* here means that we only integrate against cohomology classes pulled back from $\mathrm{H}^*(X)$, rather than allowing arbitrary classes from $\mathrm{H}^*(Y)$.

Proof. The idea, of course, is to repeatedly apply the recursion formula. The proof is by induction, and in order for the argument to work it is essential that we determine simultaneously the absolute invariants of Y and the relative invariants of (X, Y).

We induct on: the intersection number $d = Y \cdot \beta$, the number of marked points n, and the total tangency $\Sigma_i \alpha_i$, in that order. This means that when we come to compute an absolute or relative invariant, we assume that all of the absolute *and* relative invariants with

(1) smaller d, or
(2) the same d, but smaller n, or
(3) the same d, the same n, but smaller $\Sigma_i \alpha_i$

are known. For the purposes of this ordering, we set $\Sigma_i \alpha_i = d + 1$ for any absolute invariant of Y. This means that when we come to compute such an invariant, we assume that all the relative invariants with the same d and n are known.

We first prove the induction step for the relative invariants; suppose then that we want to compute some invariant:

$$\langle \gamma_1 \psi_1^{k_1}, \ldots, \gamma_n \psi_n^{k_n} \rangle_{0,\alpha,\beta}^{X|Y}$$

We assume $\Sigma_i \alpha_i > 0$, since otherwise this is just an absolute invariant of X. Pick some $k \in \{1, \ldots, n\}$ with $\alpha_k > 0$, and apply Theorem 2.4.1 to obtain:

$$((\alpha_k - 1)\psi_k + \mathrm{ev}_k^*[Y]) \cap [\mathcal{Q}_{0,\alpha-e_k}(X|Y,\beta)]^{\mathrm{virt}} = [\mathcal{Q}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}} + [\mathcal{D}^{\mathcal{Q}}_{\alpha-e_k,k}(X|Y,\beta)]^{\mathrm{virt}}$$

Capping this with the appropriate product of evaluation and psi classes, we obtain from the first term on the right-hand side the invariant that we are looking for.

It remains to show that the other terms are known by the induction hypothesis. Clearly, this is true for the term on the left-hand side, which has the same d, the same n, but smaller $\Sigma_i \alpha_i$. Consider on the other hand a component of the comb locus. This contributes a product of an absolute invariant of Y (corresponding to the internal component) with a number of relative invariants of (X, Y) (corresponding to the external components). One can check that each of these invariants either has smaller d, or the same d and smaller n. Thus, they are also determined. Therefore the relative invariant is determined inductively.

Now we prove the induction step for the absolute invariants of Y. Suppose then that we want to compute a restricted invariant:

$$\langle \gamma_1 \psi_1^{k_1}, \ldots, \gamma_n \psi_n^{k_n} \rangle_{0,n,\beta}^{Y}$$

If we apply Theorem 2.4.1 with $\alpha = (d+1, 0, \ldots, 0)$ we obtain

$$(d\psi_1 + \mathrm{ev}_1^*[Y]) \cap [\mathcal{Q}_{0,\alpha - e_1}(X|Y,\beta)]^{\mathrm{virt}} = [\mathcal{D}_{\alpha,1}^{\mathcal{Q}}(X|Y,\beta)]^{\mathrm{virt}}$$

where the comb locus on the right-hand side has a connected component isomorphic to the moduli space

$$\mathcal{Q}_{0,n}(Y,\beta)$$

(corresponding to a "comb with no teeth"). Capping as before with an appropriate class, we obtain the invariant that we are looking for. The term on the left-hand side is known since $\Sigma_i \alpha_i$ is smaller, while any other terms coming from the comb locus either involve invariants with smaller d or with the same d but smaller n, and so are also known inductively. This completes the proof. □

Remark 2.5.1. There is a subtle but extremely important point which we have ignored in the proof above. While the statement of Theorem 2.5.1 only concerns the *restricted* quasimap invariants, i.e. those with insertions from $\mathrm{H}^*(X)$, when we calculate contributions from the comb loci we are forced to consider unrestricted invariants, due to classes in the diagonal in $\mathrm{H}^*(Y \times Y)$ which do not come from $\mathrm{H}^*(X \times X)$. This is problematic, since in general these terms cannot be computed inductively.

However, a careful analysis of the recursion formula shows that any term which appears in this way must in fact be zero. The argument is the same as the one given for Gromov–Witten invariants in [Gat03a, §2.5]; the details are left to the reader. The key idea is to show that any absolute or relative quasimap invariant which has precisely one insertion from outside of

$\mathrm{H}^*(X)$ must be zero, and then to show that any term arising from the comb locus and involving unrestricted classes is equal to a product of invariants, at least one of which takes this form.

2.5.2. **A mirror theorem for quasimap invariants.** Although the algorithm presented in the previous section is completely explicit, it is in general quite involved, since the combinatorics can become arbitrarily complicated. We would like to be able to find a closed formula which expresses the quasimap invariants of Y in terms of those of X. This is our goal over the next few sections, culminating in Theorem 2.5.2, which provides such a closed formula, under some additional restrictions.

In [Gat03b] Gathmann applies the stable map recursion formula to obtain a new proof of the mirror theorem for hypersurfaces [Giv96]. This can be viewed as a partial quantum Lefschetz formula, expressing certain stable map invariants of Y in terms of those of X.

In this section we carry out a similar computation in the quasimap setting. We work with generating functions for 2-pointed quasimap invariants (the minimal number of markings, due to the strong stability condition). The absence of rational tails in the quasimap moduli space makes the quasimap recursion much simpler than Gathmann's.

Our formula can be viewed as a special case of [CFK14, Corollary 5.5.1], and thus as a relation between certain residues of the $\mathbb{G}_{\mathrm{m}}$-action on spaces of 0-pointed and 1-pointed parametrised quasimaps to Y. Some of the consequences of this formula are explored in [CFK14, Section 5.5]; for instance, it follows in the semipositive case that all primary ϵ-quasimap invariants with a fundamental class insertion can be expressed in terms of 2-pointed invariants.

2.5.3. **Setup.** As before, we let $X = X_\Sigma$ be a smooth projective toric variety and $i\colon Y \hookrightarrow X$ a smooth very ample hypersurface. We also make the following two assumptions:

(1) *Y is semi-positive*: $-K_Y$ is nef;
(2) *Y contains all curve classes*: the map $i_* : \mathrm{A}_1(Y) \to \mathrm{A}_1(X)$ is surjective.

By adjunction, $-K_X$ pairs strictly positively with every curve class coming from Y, hence with every curve class by Assumption (2). Thus $-K_X$ is ample, or in other words, X is Fano[3].

[3]Kleiman's criterion says that a divisor D is ample if and only if $D \cdot C > 0$ for every curve class C in the closure of the effective cone. But since X is a toric variety the effective cone is finitely generated in $\mathrm{A}_1(X)$, hence is closed in $\mathrm{A}_1(X)_{\mathbb{R}}$ as it is a finite intersection of half-spaces. So we only need to check $D \cdot C > 0$ for every effective curve class.

Also note that if $\dim X \geq 3$ then Assumption (2) always holds, due to the classical Lefschetz hyperplane theorem; on the other hand if $\dim X = 2$ then Assumption (2) forces X to be $\mathbb{P}^2$.

We fix a homogeneous basis $\eta_0, \ldots, \eta_k$ for $\mathrm{H}^*(X) = \mathrm{H}^*(X, \mathbb{Q})$ and let $\eta^0, \ldots, \eta^k$ denote the dual basis with respect to the Poincaré pairing. Without loss of generality we may suppose that $\eta^0 = \mathbb{1}_X$ and $\eta^1 = [Y]$. We get an induced basis $\rho_1 = i^*\eta_1, \ldots, \rho_k = i^*\eta_k$ for $i^* \mathrm{H}^*(X)$. Notice that $\rho_0 = i^*\eta_0 = i^*[\mathrm{pt}_X] = 0$, $\rho_1 = i^*\eta_1 = [\mathrm{pt}_Y]$. We can extend the ρ_i to a basis $\rho_1, \ldots, \rho_l$ for $\mathrm{H}^*(Y)$ by adding $\rho_{k+1} \ldots, \rho_l$. Let $\rho^1, \ldots, \rho^l$ denote the dual basis; notice that ρ^i is *not* equal to $i^*\eta^i$ (they do not even have the same degree!). Note also that $\rho^1 = \mathbb{1}_Y$.

2.5.4. **Generating functions for quasimap invariants**. As with many results in enumerative geometry, the quasimap Lefchetz formula is most conveniently stated in terms of generating functions. Here we define several such generating functions for the absolute quasimap invariants of X and Y. We work with two marked points since this is the minimum number required in order for the quasimap space to be nonempty. However since we only take insertions at the first marking we would like to think of these, morally speaking, as 1-pointed invariants (in Gromov–Witten theory the corresponding statement is literally true, due to the string equation).

For any smooth projective toric variety[4] X and any effective curve class $\beta \in \mathrm{H}_2^+(X)$, we define

$$S_0^X(z, \beta) = (\mathrm{ev}_1)_* \left(\frac{1}{z - \psi_1} [\mathcal{Q}_{0,2}(X, \beta)]^{\mathrm{virt}} \right)$$

and

$$S_0^X(z, q) = \sum_{\beta \geq 0} q^\beta S_0^X(z, \beta)$$

where by convention $S_0^X(z, \beta) = \mathbb{1}_X$ for $\beta = 0$, and q is a Novikov variable. These are generating functions for quasimap invariants of X which take values in $\mathrm{H}^*(X)$.

The same definition applies to Y. However, sometimes we may wish to consider only insertions of cohomology classes coming from X. These are the so-called *restricted quasimap invariants*, and the corresponding generating function is defined as

$$\tilde{S}_0^Y(z, \beta) = (\mathrm{ev}_1)_* \left(\frac{1}{z - \psi_1} [\mathcal{Q}_{0,2}(Y, \beta)]^{\mathrm{virt}} \right)$$

where crucially ev_1 is viewed as *mapping to* X instead of to Y. Thus $\tilde{S}_0^Y(z, \beta)$ takes values in $\mathrm{H}^*(X)$ and involves only quasimap invariants of Y with insertions coming from $i^* \mathrm{H}^*(X)$; this

[4] Or more generally any space for which the quasimap invariants are defined, for instance a smooth hypersurface in a toric variety.

is in contrast to $S_0^Y(z,\beta)$, which takes values in $\mathrm{H}^*(Y)$ and involves quasimap invariants of Y with arbitrary insertions. As earlier, we can also define $\tilde{S}_0^Y(z,q)$.

Now, since X and Y are smooth, we may use Poincaré duality to define a push-forward map on cohomology, $i_* \colon \mathrm{H}^k(Y) \to \mathrm{H}^{k+2}(X)$.

Lemma 2.5.2. $i_* S_0^Y(z,\beta) = \tilde{S}_0^Y(z,\beta)$.

Proof. This follows from functoriality of cohomological push-forwards and the fact that we have a commuting triangle:

$$\begin{array}{ccc} \mathcal{Q}_{0,2}(Y,\beta) & \xrightarrow{\ \mathrm{ev}_1\ } & Y \\ & {\scriptstyle \mathrm{ev}_1} \searrow \quad \swarrow {\scriptstyle i} & \\ & X & \end{array}$$

Let us spell this out explicitly, in order to help familiarise the reader with the generating functions involved. First, it is easy to see from the projection formula that:

$$i_*\rho^i = \begin{cases} \eta^i & \text{for } i = 1, \ldots, k \\ 0 & \text{for } i = k+1, \ldots, l \end{cases}$$

Now, we can write $S_0^Y(z,\beta)$ as:

$$S_0^Y(z,\beta) = \sum_{i=1}^{l} \left\langle \frac{\rho_i}{z - \psi_1}, \mathbb{1}_Y \right\rangle_{0,2,\beta}^{Y} \rho^i$$

Thus applying i_* gives

$$i_* S_0^Y(z,\beta) = \sum_{i=1}^{l} \left\langle \frac{\rho_i}{z - \psi_1}, \mathbb{1}_Y \right\rangle_{0,2,\beta}^{Y} i_*\rho^i = \sum_{i=1}^{k} \left\langle \frac{\eta_i}{z - \psi_1}, \mathbb{1}_X \right\rangle_{0,2,\beta}^{Y} \eta^i = \tilde{S}_0^Y(z,\beta)$$

as claimed. □

2.5.5. **Quasimap Lefschetz formula.** We now turn to our main result: a formula expressing the generating function $\tilde{S}_0^Y(z,q)$ for restricted quasimap invariants of Y in terms of the quasimap invariants of X.

Theorem 2.5.2. Let X and Y be as above. Then

$$\frac{\sum_{\beta \geq 0} q^\beta \prod_{j=0}^{Y\cdot\beta} (Y + jz) S_0^X(z,\beta)}{P_0^X(q)} = \tilde{S}_0^Y(z,q) \tag{2.5.1}$$

where:

$$P_0^X(q) = 1 + \sum_{\substack{\beta>0 \\ K_Y\cdot\beta=0}} q^\beta (Y\cdot\beta)! \langle [\mathrm{pt}_X]\psi_1^{Y\cdot\beta-1}, \mathbb{1}_X\rangle^X_{0,2,\beta}$$

Notice that $P_0^X(q)$ depends not only on X but also on the divisor class of Y in X; the superscript is supposed to indicate that the definition only involves quasimap invariants of X.

Proof. For $m = 0, \ldots, Y\cdot\beta$, define the following generating function for 2-pointed relative quasimap invariants

$$S^{X|Y}_{0,(m)}(z,\beta) = (\mathrm{ev}_1)_* \left(\frac{1}{z-\psi_1} [\mathcal{Q}_{0,(m,0)}(X|Y,\beta)]^{\mathrm{virt}} \right)$$

where we view ev_1 as mapping to X. Note that $S^{X|Y}_{0,(0)}(z,\beta) = S^X_0(z,\beta)$. Also define the following generating function for "comb loci invariants"

$$T^{X|Y}_{0,(m)}(z,\beta) = (\mathrm{ev}_1)_* \left(m[\mathcal{Q}_{0,(m,0)}(X|Y,\beta)]^{\mathrm{virt}} + \frac{1}{z-\psi_1}[\mathcal{D}^{\mathcal{Q}}_{(m,0),1}(X|Y,\beta)]^{\mathrm{virt}} \right)$$

where again we view ev_1 as mapping to X. As in [Gat03b, Lemma 1.2], it follows from Theorem 2.4.1 that

$$(Y+mz)S^{X|Y}_{0,(m)}(z,\beta) = S^{X|Y}_{0,(m+1)}(z,\beta) + T^{X|Y}_{0,(m)}(z,\beta) \tag{2.5.2}$$

and we can apply this repeatedly to obtain:

$$\prod_{j=0}^{Y\cdot\beta}(Y+jz)S^X_0(z,\beta) = \sum_{m=0}^{Y\cdot\beta}\prod_{j=m+1}^{Y\cdot\beta}(Y+jz)T^{X|Y}_{0,(m)}(z,\beta) \tag{2.5.3}$$

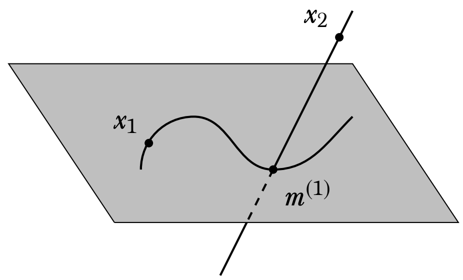
We now examine the right-hand side in detail. By definition, $T^{X|Y}_{0,(m)}(z,\beta)$ splits into two parts: those terms coming from the relative space and those terms coming from the comb loci.

Let us first consider the contribution of the comb loci. Since there are only two marked points and the first is required to lie on the internal component of the comb, it follows from the strong stability condition that there are only two options: a comb with zero teeth or a comb with one tooth.

First consider the case of a comb with zero teeth. The moduli space is then $\mathcal{Q}_{0,2}(Y,\beta)$ and we require that $Y\cdot\beta = m$. Thus this piece only contributes to $T^{X|Y}_{0,(Y\cdot\beta)}(z,\beta)$, and the contribution is:

$$\sum_{i=1}^{k} \left\langle \frac{\rho_i}{z-\psi_1}, \mathbb{1}_Y \right\rangle^Y_{0,2,\beta} \eta^i$$

Next consider the case of a comb with one tooth. Let $\beta^{(0)}$ and $\beta^{(1)}$ denote the curve classes of the internal and external components, respectively, and let $m^{(1)}$ be the contact order of the external component with Y. The picture is as follows



and the invariants which contribute take the form

$$\left\langle \frac{\rho_i}{z-\psi_1}, \rho^h \right\rangle^Y_{0,2,\beta^{(0)}} \left\langle \rho_h, \mathbb{1}_X \right\rangle^{X|Y}_{0,(m^{(1)},0),\beta^{(1)}}$$

for $i = 1,\ldots,k$ and $h = 1,\ldots,l$. By computing dimensions, we find

$$\begin{aligned} 0 \leq \operatorname{codim} \rho^h &= \dim Y - \operatorname{codim} \rho_h \\ &= \dim Y - \operatorname{vdim} \mathcal{Q}_{0,(m^{(1)},0)}(X|Y,\beta^{(1)}) \\ &= \dim Y - (\dim X - 3 - K_X \cdot \beta^{(1)} + 2 - m^{(1)}) \\ &= K_Y \cdot \beta^{(1)} - Y \cdot \beta^{(1)} + m^{(1)} \\ &\leq 0 \end{aligned}$$

where the final equality follows from adjunction and the final inequality holds because $-K_Y$ is nef and $m^{(1)} \leq Y \cdot \beta_1$. This shows that the only non-trivial contributions come from curve classes $\beta^{(1)}$ such that $K_Y \cdot \beta^{(1)} = 0$, and that in this case the order of tangency must be maximal, i.e. $m^{(1)} = Y \cdot \beta^{(1)}$. Furthermore we must have $\operatorname{codim} \rho^h = 0$ and so $\rho^h = \rho^1 = \mathbb{1}_Y$ which implies $\rho_h = \rho_1 = [\mathrm{pt}_Y]$. Finally since $m^{(1)} = Y \cdot \beta^{(1)}$ we have

$$m = Y \cdot \beta^{(0)} + m^{(1)} = Y \cdot (\beta^{(0)} + \beta^{(1)}) = Y \cdot \beta$$

and so again this piece only contributes to $T^{X|Y}_{0,(Y\cdot\beta)}(z,\beta)$, and the contribution is:

$$\sum_{i=1}^k \left(\sum_{\substack{0<\beta^{(1)}<\beta \\ K_Y\cdot\beta^{(1)}=0}} (Y\cdot\beta^{(1)}) \left\langle \frac{\rho_i}{z-\psi_1}, \mathbb{1}_Y \right\rangle^Y_{0,2,\beta-\beta^{(1)}} \left\langle \rho_1, \mathbb{1}_X \right\rangle^{X|Y}_{0,(Y\cdot\beta^{(1)},0),\beta^{(1)}} \right) \eta^i$$

where the $Y\cdot\beta^{(1)}$ factor comes from the weighting on the virtual class of the comb locus. Finally, we must examine the terms of $T^{X|Y}_{0,(m)}(z,\beta)$ coming from:

$$\mathrm{ev}_{1*}(m[\mathcal{Q}_{0,(m,0)}(X|Y,\beta)]^{\mathrm{virt}})$$

Notice that we only have insertions from $i^*\,\mathrm{H}^*(X)\subseteq \mathrm{H}^*(Y)$, since ev_1 is viewed as mapping to X. On the other hand

$$\begin{aligned}
\operatorname{vdim}\mathcal{Q}_{0,(m,0)}(X|Y,\beta) &= \dim X - 3 - K_X\cdot\beta + 2 - m \\
&= \dim X - 1 - K_Y\cdot\beta + Y\cdot\beta - m && \text{by adjunction} \\
&\geq \dim X - 1 + Y\cdot\beta - m && \text{since } -K_Y \text{ is nef} \\
&\geq \dim X - 1 && \text{since } m\leq Y\cdot\beta
\end{aligned}$$

where in the second line we have applied the projection formula to i, and thus have implicitly used Assumption (2), discussed in §2.5.3; namely that every curve class on X comes from a class on Y.

Consequently the only insertions that can appear are those of dimension 0 and 1. However, the restriction of the 0-dimensional class $\eta_0=[\mathrm{pt}_X]$ to Y vanishes, as do the restrictions of all 1-dimensional classes except for η_1 (by the definition of the dual basis, since $\eta^1=Y$). Thus the only insertion is $i^*\eta_1=\rho_1=[\mathrm{pt}_Y]$, and since η^1 has dimension 1 all the inequalities above must actually be equalities. Thus we only have a contribution if $-K_Y\cdot\beta=0$ and $m=Y\cdot\beta$. The contribution to $T^{X|Y}_{0,(Y\cdot\beta)}(z,\beta)$ in this case is:

$$(Y\cdot\beta)\langle\rho_1,\mathbb{1}_X\rangle^{X|Y}_{0,(Y\cdot\beta,0),\beta}\eta^1$$

Thus we have calculated $T^{X|Y}_{0,(m)}(z,\beta)$ for all m; substituting into equation (2.5.3) we obtain

$$\begin{aligned}
\prod_{j=0}^{Y\cdot\beta}(Y+jz)S_0^X(z,\beta) &= T^{X|Y}_{0,(Y\cdot\beta)}(z,\beta) \\
&= \sum_{i=1}^{k}\left\langle\frac{\rho_i}{z-\psi_1},\mathbb{1}_Y\right\rangle^{Y}_{0,2,\beta}\eta^i+ \\
&\quad \sum_{i=1}^{k}\left(\sum_{\substack{0<\beta^{(1)}<\beta\\ K_Y\cdot\beta^{(1)}=0}}(Y\cdot\beta^{(1)})\left\langle\frac{\rho_i}{z-\psi_1},\mathbb{1}_Y\right\rangle^{Y}_{0,2,\beta-\beta^{(1)}}\left\langle\rho_1,\mathbb{1}_X\right\rangle^{X|Y}_{0,(Y\cdot\beta^{(1)},0),\beta^{(1)}}\right)\eta^i+ \\
&\quad (Y\cdot\beta)\langle\rho_1,\mathbb{1}_X\rangle^{X|Y}_{0,(Y\cdot\beta,0),\beta}\eta^1
\end{aligned}$$

where the third term only appears if $K_Y \cdot \beta = 0$. We can rewrite this as:

$$\prod_{j=0}^{Y\cdot\beta}(Y+jz)S_0^X(z,\beta) = \tilde{S}_0^Y(z,\beta) + \sum_{\substack{0<\beta^{(1)}\leq\beta \\ K_Y\cdot\beta^{(1)}=0}} \left((Y\cdot\beta^{(1)}) \left\langle \rho_1, \mathbb{1}_X \right\rangle^{X|Y}_{0,(Y\cdot\beta^{(1)},0),\beta^{(1)}} \right) \tilde{S}_0^Y(z,\beta-\beta^{(1)})$$

It is now clear from the expression above that equation (2.5.1) in the statement of Theorem 2.5.2 holds, with:

$$P_0^X(q) = 1 + \sum_{\substack{\beta>0 \\ K_Y\cdot\beta=0}} q^\beta (Y\cdot\beta)\langle \rho_1, \mathbb{1}_X\rangle^{X|Y}_{0,(Y\cdot\beta,0),\beta}$$

To complete the proof it thus remains to show that:

$$P_0^X(q) = 1 + \sum_{\substack{\beta>0 \\ K_Y\cdot\beta=0}} q^\beta (Y\cdot\beta)!\langle \psi_1^{Y\cdot\beta-1}[\mathrm{pt}_X], \mathbb{1}_X\rangle^X_{0,2,\beta}$$

The aim therefore is to express the relative invariants

$$\langle \rho_1, \mathbb{1}_X\rangle^{X|Y}_{0,(Y\cdot\beta,0),\beta}$$

in terms of absolute invariants of X. Unsurprisingly, we once again do this by applying Theorem 2.4.1. We have:

$$[\mathcal{Q}_{0,(Y\cdot\beta,0)}(X|Y,\beta)]^{\mathrm{virt}} = ((Y\cdot\beta-1)\psi_1 + \mathrm{ev}_1^* Y)[\mathcal{Q}_{0,(Y\cdot\beta-1,0)}(X|Y,\beta)]^{\mathrm{virt}} - [\mathcal{D}^{\mathcal{Q}}_{(Y\cdot\beta-1,0),1}(X|Y,\beta)]^{\mathrm{virt}}$$

We begin by examining the contributions from the comb loci. As before, we have only contributions coming from combs with 0 teeth and combs with 1 tooth. The former contributions take the form

$$\langle \rho_1, \mathbb{1}_Y\rangle^Y_{0,2,\beta}$$

which vanish because $\mathrm{vdim}\,\mathcal{Q}_{0,2}(Y,\beta) = \dim Y - 1 - K_Y\cdot\beta = \dim Y - 1$ whereas the insertion has codimension $\dim Y$. The latter contributions take the form

$$\langle \rho_1, \rho^h\rangle^Y_{0,2,\beta^{(0)}}\langle \rho_h, \mathbb{1}_X\rangle^{X|Y}_{0,(Y\cdot(\beta-\beta^{(0)})-1,0),\beta-\beta^{(0)}}$$

and these must also vanish since:

$$\begin{aligned}
\operatorname{codim} \rho^h &= \dim Y - \operatorname{codim} \rho_h \\
&= \dim Y - \operatorname{vdim} \mathcal{Q}_{0,(Y\cdot(\beta-\beta^{(0)})-1,0)}(X|Y, \beta - \beta^{(0)}) \\
&= \dim Y - (\dim X - 3 - K_X \cdot (\beta - \beta^{(0)}) + 2 - Y \cdot (\beta - \beta^{(0)}) + 1) \\
&= -1 + K_X \cdot (\beta - \beta^{(0)}) + Y \cdot (\beta - \beta^{(0)}) \\
&= -1 + K_Y \cdot (\beta - \beta^{(0)}) \\
&\leq -1
\end{aligned}$$

Thus the comb loci do not contribute at all. Applying this recursively (the same argument as above shows that we never get comb loci contributions), we find that

$$\begin{aligned}
(Y \cdot \beta)\langle \rho_1, \mathbb{1}_X \rangle^{X|Y}_{0,(Y\cdot\beta,0),\beta} &= (Y \cdot \beta)\langle \eta_1 \prod_{j=0}^{Y\cdot\beta-1} (Y + j\psi_1), \mathbb{1}_X \rangle^X_{0,2,\beta} \\
&= (Y \cdot \beta)! \langle [\mathrm{pt}_X] \psi_1^{Y\cdot\beta-1}, \mathbb{1}_X \rangle^X_{0,2,\beta}
\end{aligned}$$

where the second equality holds because $Y \cdot \eta_1 = \eta^1 \cdot \eta_1 = [\mathrm{pt}_X]$ and $Y^2 \cdot \eta_1 = 0$. This completes the proof of Theorem 2.5.2. □

Corollary 2.5.3. If Y is Fano then there is no correction term:

$$\sum_{\beta \geq 0} q^\beta \prod_{j=0}^{Y\cdot\beta} (Y + jz) S_0^X(z, \beta) = \tilde{S}_0^Y(z, q)$$

Corollary 2.5.4. Let $Y = Y_5 \subseteq X = \mathbb{P}^4$ be the quintic three-fold. Then

$$\tilde{S}_0^{Y_5}(z, q) = \frac{I_{\mathrm{sm}}^{Y_5}(z, q)}{P(q)}$$

where

$$I_{\mathrm{sm}}^{Y_5}(z, q) = 5H + \sum_{d>0} \frac{\prod_{j=0}^{5d} (H + jz)}{\prod_{j=0}^{d} (H + jz)^5} \, q^d$$

and:

$$P(q) = 1 + \sum_{d>0} \frac{(5d)!}{(d!)^5} q^d$$

Proof. Apply Theorem 2.5.2 and use the fact that the quasimap invariants of $\mathbb{P}^4$ coincide with the Gromov–Witten invariants, which are well-known from mirror symmetry. □

Remark 2.5.5. Theorem 2.5.2 agrees with [CZ14, Theorem 1] when X is a projective space.

2.5.6. **Comparison with the work of Ciocan-Fontanine and Kim**. Here we briefly explain how to compare our Theorem 2.5.2 to a formula obtained by Ciocan-Fontanine and Kim. We assume that the reader is familiar with [CFK14], in particular §4 and §5 thereof. There they introduce (in the more general context of ϵ-stable quasimaps) the following generating functions for quasimap invariants of Y:

(1) The *J^ϵ-function*

$$J^\epsilon(\mathbf{t}, z) = \sum_{m \geq 0, \beta \geq 0} \frac{q^\beta}{m!} (\mathrm{ev}_\bullet)_* \left(\prod_{i=1}^m \mathrm{ev}_i^*(\mathbf{t}) \cap \mathrm{Res}_{F_0} [\mathcal{QG}^\epsilon_{0,m}(Y, \beta)]^{\mathrm{virt}} \right)$$

for $\mathbf{t} \in \mathrm{H}^*(Y)$. Here $\mathcal{QG}^\epsilon_{0,m}(Y, \beta)$ is the moduli space of ϵ-stable quasimaps with a parametrised component, F_0 is a certain fixed locus of the natural $\mathbb{G}_\mathrm{m}$-action on this space, and $\mathrm{ev}_\bullet$ is the evaluation at the point $\infty \in \mathbb{P}^1$ on the parametrised component. Res_{F_0} is the residue of the virtual class, i.e. the virtual class of the fixed locus divided by the Euler class of the virtual normal bundle (see [GP99] for details on virtual localisation). The variable z is the $\mathbb{G}_\mathrm{m}$-equivariant parameter.

(2) The *S^ϵ-operator*

$$S^\epsilon(\mathbf{t}, z)(\gamma) = \sum_{m \geq 0, \beta \geq 0} \frac{q^\beta}{m!} (\mathrm{ev}_1)_* \left(\frac{\mathrm{ev}_2^*(\gamma) \cdot \prod_{j=3}^{2+m} \mathrm{ev}_j^*(\mathbf{t})}{z - \psi_1} \cap [\mathcal{Q}^\epsilon_{0,2+m}(Y, \beta)]^{\mathrm{virt}} \right)$$

where $\mathbf{t}, \gamma \in \mathrm{H}^*(X)$ and z is a formal variable. This is the quasimap analogue of the fundamental solution matrix; see §4.2.3.

(3) The *P^ϵ-series*

$$P^\epsilon(\mathbf{t}, z) = \sum_{h=1}^k \rho^h \sum_{m \geq 0, \beta \geq 0} \frac{q^\beta}{m!} \left(\mathrm{ev}_1^*(\rho_h \boxtimes p_\infty) \cap [\mathcal{QG}^\epsilon_{0,1+m}(Y, \beta)]^{\mathrm{virt}} \right)$$

where $\mathbf{t} \in \mathrm{H}^*(X)$ and z is the $\mathbb{G}_\mathrm{m}$-equivariant parameter. Here we view ev_1 as mapping to $Y \times \mathbb{P}^1$, and $p_\infty \in \mathrm{H}^*_{\mathbb{G}_\mathrm{m}}(\mathbb{P}^1)$ is the equivariant cohomology class defined by setting $p_\infty|_0 = 0$ and $p_\infty|_\infty = -z$. If we present the equivariant cohomology ring as $\mathrm{H}^*_{\mathbb{G}_\mathrm{m}}(\mathbb{P}^1) = \Bbbk[H, z]/(H^2 - z^2)$, then $p_\infty = (H - z)/2$.

Given these definitions, Ciocan-Fontanine and Kim use localisation with respect to the $\mathbb{G}_\mathrm{m}$-action on the parametrised space to prove the following formula [CFK14, Theorem 5.4.1]:

$$J^\epsilon(\mathbf{t}, z) = S^\epsilon(\mathbf{t}, z)(P^\epsilon(\mathbf{t}, z))$$

They observe (via a dimension count) that if we set $\mathbf{t} = 0$ and restrict to semi-positive targets, then the only class that matches non-trivially with $P^{\epsilon}|_{\mathbf{t}=0}$ is $[\mathrm{pt}_Y]$. Hence the above formula takes the simple form

$$\frac{J^{\epsilon}|_{\mathbf{t}=0}}{\langle [\mathrm{pt}_Y], P^{\epsilon}|_{\mathbf{t}=0}\rangle} = S^{\epsilon}(\mathbb{1}_Y)|_{\mathbf{t}=0} = \mathbb{1}_Y + \sum_{h=1}^{k} \rho^h \left(\sum_{\beta > 0} q^{\beta} \left\langle \frac{\rho_h}{z-\psi}, \mathbb{1}_Y \right\rangle_{0,2,\beta}^{Y,\epsilon} \right) \tag{2.5.4}$$

see [CFK14, Corollary 5.5.1]. In our setting, $\epsilon = 0+$ and Y embeds as a very ample hypersurface in a toric Fano variety X. Our Theorem 2.5.2 makes explicit a consequence of formula (2.5.4). More precisely:

Lemma 2.5.6. We have the following relations between our generating functions and the generating functions of Ciocan-Fontanine and Kim:

$$i_* J^{0+}|_{\mathbf{t}=0} = \sum_{\beta \geq 0} q^{\beta} \prod_{j=0}^{Y\cdot\beta} (Y + jz) S_0^X(z,\beta) \tag{2.5.5}$$

$$\langle [\mathrm{pt}_Y], P^{0+}|_{\mathbf{t}=0}\rangle = P_0^X(q) \tag{2.5.6}$$

$$i_* S^{0+}(\mathbb{1}_Y)|_{\mathbf{t}=0} = \tilde{S}_0^Y(z,q) \tag{2.5.7}$$

Proof. (2.5.7) is clear from the second equality of (2.5.4) and the definition of $\tilde{S}_0^Y(z,q)$. To show (2.5.5), let us look more closely at the left-hand side:

$$J^{0+}|_{\mathbf{t}=0} = \sum_{\beta \geq 0} q^{\beta} (\mathrm{ev}_{\bullet})_* \left(\mathrm{Res}_{F_0} [\mathcal{Q}G_{0,0}(Y,\beta)]^{\mathrm{virt}} \right)$$

We have a diagram of fixed loci and evaluation maps

$$\begin{array}{ccccc} \mathcal{Q}G_{0,0}(Y,\beta) & \longleftarrow & F_0^Y & \xrightarrow{\mathrm{ev}_\bullet} & Y \\ \downarrow{\scriptstyle i} & \square & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ \mathcal{Q}G_{0,0}(X,\beta) & \longleftarrow & F_0^X & \xrightarrow{\mathrm{ev}_\bullet} & X \end{array}$$

and by a mild generalisation of [CFKM14, Propositions 6.2.2 and 6.2.3], we have an equality of $\mathbb{G}_{\mathrm{m}}$-equivariant classes

$$i_*[\mathcal{Q}G_{0,0}(Y,\beta)]^{\mathrm{virt}} = e(\pi_* E^Y_{0,0,\beta}) \cap [\mathcal{Q}G_{0,0}(X,\beta)]^{\mathrm{virt}}$$

where π is the universal curve on $\mathcal{Q}G_{0,0}(X,\beta)$ and $E^Y_{0,0,\beta}$ is the equivariant line bundle[5] on this curve associated to $\mathcal{O}_X(Y)$.

[5] This is the parametrised analogue of the bundle L_Y constructed in the definition of relative quasimaps; see §2.2.3.

We would like to pull back this equation to the fixed locus F_0^X in order to obtain an equation involving the residues. Let us first briefly recall the definition of F_0^X. Since there are no markings, any quasimap in $\mathcal{Q}G_{0,0}(X,\beta)$ has irreducible source curve. For such a quasimap to be $\mathbb{G}_{\mathrm{m}}$-fixed we need that the induced rational map is constant; this means that the degree of the quasimap is concentrated at the basepoints (i.e. the sum of the lengths of the basepoints should be equal to the degree). Furthermore only the points 0 and ∞ of the parametrised component are allowed to be basepoints. The fixed loci are thus indexed by ordered partitions of the degree which record the length of the basepoints at 0 and ∞. F_0^X is the locus on which all the degree is concentrated at 0. This means that ∞ is not a basepoint and we have an evaluation map ev_∞ (denoted $\mathrm{ev}_\bullet$ earlier). See [CFK14, §4] for more details: our F_0^X is there denoted $F_{0,0,\beta}^{0,0,0}$.

Since the fibres of π are irreducible and rational, the degree of the universal line bundle on the parametrised component is constant; therefore we have for $0 < j \leq Y \cdot \beta + 1$ an exact sequence:

$$0 \to \pi_*(E_{0,0,\beta}^Y(-j\sigma_\infty)) \to \pi_* E_{0,0,\beta}^Y \to \sigma_\infty^* \mathcal{P}^{j-1}(E_{0,0,\beta}^Y) \to 0$$

where $\mathcal{P}^{j-1}$ denotes the bundle of $(j-1)$-jets, and σ_∞ is the section given by the point $\infty \in \mathbb{P}^1$ of the parametrised component. The right-hand map is given by evaluating a section of $E_{0,0,\beta}^Y$ (as well as its derivatives up to order $j-1$) at the point ∞. The left-hand term consists of sections of $E_{0,0,\beta}^Y$ which vanish at σ_∞ to order j. If we set $j = Y \cdot \beta + 1$ then this term vanishes and we have:

$$\pi_* E_{0,0,\beta}^Y = \sigma_\infty^* \mathcal{P}^{Y \cdot \beta}(E_{0,0,\beta}^Y)$$

On the other hand, we have

$$0 \to E_{0,0,\beta}^Y \otimes \omega_\pi^{\otimes j} \to \mathcal{P}^j(E_{0,0,\beta}^Y) \to \mathcal{P}^{j-1}(E_{0,0,\beta}^Y) \to 0$$

see [Gat02, §2]. Pulling back along σ_∞ and taking Euler classes, we can compute recursively from $j = Y \cdot \beta$ to 0 and obtain a splitting

$$e(\pi_* E_{0,0,\beta}^Y) = \prod_{j=0}^{Y \cdot \beta} c_1(\sigma_\infty^* E_{0,0,\beta}^Y \otimes \omega_\infty^{\otimes j})$$

where $\omega_\infty = \sigma_\infty^* \omega_\pi$ gives the cotangent space at the point ∞. The bundle ω_∞ is (non-equivariantly) trivial since the source curves in F_0^X are rigid; on the other hand the weight of the $\mathbb{G}_{\mathrm{m}}$-action on the cotangent space at ∞ is z. We thus obtain:

$$i_*[F_0^Y]^{\mathrm{virt}} = \prod_{j=0}^{Y\cdot\beta}(\mathrm{ev}_\infty^* Y + jz) \cap [F_0^X]^{\mathrm{virt}}$$

Furthermore, the Euler classes of the virtual normal bundles match under i. Substituting into $i_* J^{0+}|_{\mathbf{t}=0}$ we find that:

$$\begin{aligned} i_* J^{0+}|_{\mathbf{t}=0} &= \sum_{\beta\geq 0} q^\beta (i \circ \mathrm{ev}_\bullet)_* \left(\mathrm{Res}_{F_0^Y}[\mathcal{Q}G_{0,0}(Y,\beta)]^{\mathrm{virt}}\right) \\ &= \sum_{\beta\geq 0} q^\beta \prod_{j=0}^{Y\cdot\beta}(Y + jz)(\mathrm{ev}_\bullet)_* \left(\mathrm{Res}_{F_0^X}[\mathcal{Q}G_{0,0}(X,\beta)]^{\mathrm{virt}}\right) \end{aligned}$$

On the other hand, if we apply (2.5.4) with X instead of Y, then the denominator on the left-hand side vanishes since X is Fano. Comparing coefficients of q^β we thus obtain

$$(\mathrm{ev}_\bullet)_* \, \mathrm{Res}_{F_0^X}[\mathcal{Q}G_{0,0}(X,\beta)]^{\mathrm{virt}} = S_0^X(z,\beta)$$

from which it follows that:

$$i_* J^{0+}|_{\mathbf{t}=0} = \sum_{\beta\geq 0} q^\beta \prod_{j=0}^{Y\cdot\beta}(Y + jz) S_0^X(z,\beta)$$

This proves (2.5.5). It remains to show (2.5.6). According to Ciocan-Fontanine and Kim, if we write the $1/z$-expansion of $J^\epsilon|_{\mathbf{t}=0}$ as

$$J^\epsilon|_{\mathbf{t}=0} = J_0^\epsilon(q)\mathbb{1}_Y + O(1/z)$$

then $\langle[\mathrm{pt}_Y], P^\epsilon|_{\mathbf{t}=0}\rangle = J_0^\epsilon(q)$. It thus remains to prove that $J_0^{0+}(q) = P_0^X(q)$.

Since X is a toric Fano variety, we have the following calculation of residues due to Givental [Giv98] (see also [CFK10, Definition 7.2.8]):

$$S_0^X(z,\beta) = \prod_{\rho\in\Sigma_X(1)} \frac{\prod_{j=-\infty}^{0}(D_\rho + jz)}{\prod_{j=-\infty}^{D_\rho\cdot\beta}(D_\rho + jz)} = \frac{\prod_{\rho:\, D_\rho\cdot\beta\leq 0}\prod_{j=D_\rho\cdot\beta}^{0}(D_\rho + jz)}{\prod_{\rho:\, D_\rho\cdot\beta>0}\prod_{j=1}^{D_\rho\cdot\beta}(D_\rho + jz)}$$

We can then apply equation (2.5.5) to find $i_* J^{0+}|_{\mathbf{t}=0}$, and hence also to find $J_0^{0+}(q)$. In the end we obtain:

$$J_0^{0+}(q) = \sum_{\beta\geq 0} q^\beta (Y\cdot\beta)! \frac{\prod_{\rho:\, D_\rho\cdot\beta<0}(-1)^{-D_\rho\cdot\beta}(-D_\rho\cdot\beta)!}{\prod_{\rho:\, D_\rho\cdot\beta>0}(D_\rho\cdot\beta)!}$$

On the other hand the coefficient

$$\langle[\mathrm{pt}_X]\psi_1^{Y\cdot\beta-1}, \mathbb{1}_X\rangle_{0,2,\beta}^X$$

which appears in our $P_0^X(q)$-series also appears in $S_0^X(z,\beta)$. So again we can find it by appealing to Givental's calculation of $S_0^X(z,q)$.

$$\begin{aligned}\langle [\mathrm{pt}_X]\psi_1^{Y\cdot\beta-1}, \mathbb{1}_X\rangle_{0,2,\beta}^X &= \mathrm{coeff}_{q^\beta z^{-Y\cdot\beta}}\langle [\mathrm{pt}_X], S_0^X(z,q)\rangle \\ &= \frac{\prod_{\rho\colon D_\rho\cdot\beta<0}(-1)^{-D_\rho\cdot\beta}(-D_\rho\cdot\beta)!}{\prod_{\rho\colon D_\rho\cdot\beta>0}(D_\rho\cdot\beta)!}\end{aligned}$$

which proves (2.5.6). We thus conclude that (2.5.4) implies our Theorem 2.5.2. □

Appendix 2.6: Notes on quasimaps

In this appendix we collect several foundational results in quasimap theory, including:

(1) *Functoriality* (§2.6.1): given a morphism $f\colon Y\to X$ we describe the induced map:

$$\mathcal{Q}(f)\colon \mathcal{Q}_{g,n}(Y,\beta)\to\mathcal{Q}_{g,n}(X,f_*\beta)$$

We also discuss (§2.6.2) when $\mathcal{Q}(f)$ admits a compatible perfect obstruction theory.

(2) *Splitting axiom* (§2.6.3): this gives an equality between two natural virtual classes on boundary strata (i.e. loci where the underlying curve is reducible of a prescribed type).

(3) *Comparison with the GIT construction* (§2.6.4): we show that for a (not necessarily toric) hypersurface $Y\hookrightarrow X$, our definition of $\mathcal{Q}(Y)$ as a substack of $\mathcal{Q}(X)$ coincides with the definition of $\mathcal{Q}(Y)$ given by the description of Y as a GIT quotient (see [CFKM14]).

2.6.1. **Functoriality**. In the case of stable maps, a morphism $f: Y\to X$ induces a morphism between the corresponding moduli spaces

$$\mathcal{M}(f):\overline{\mathcal{M}}_{g,n}(Y,\beta)\to\overline{\mathcal{M}}_{g,n}(X,f_*\beta)$$

given by post-composition with f and (if necessary) stabilisation of the source curve. Because of this, we may say that the construction of the moduli space of stable maps is *functorial*.

It is natural to ask whether the same holds for the moduli space of quasimaps, i.e. whether we have a morphism:

$$\mathcal{Q}(f):\mathcal{Q}_{g,n}(Y,\beta)\to\mathcal{Q}_{g,n}(X,f_*\beta)$$

Since here the objects of the moduli space are not maps, we cannot simply compose with f. Nevertheless, our definition should be equivalent to composing with f when applied to a quasimap without any basepoints. In [CFK14, Section 3.1] a definition (in the GIT context) is given when f is an embedding into a projective space; we shall discuss the general situation of a morphism between toric varieties $f\colon Y\to X$.

Our approach uses the language of Σ–collections introduced by D. A. Cox. This approach is natural insofar as a quasimap is a generalisation of a Σ–collection. We will refer extensively to [Cox95b] and [Cox95a].

Let X and Y be smooth and proper toric varieties with fans $\Sigma_X \subseteq N_X$ and $\Sigma_Y \subseteq N_Y$. Suppose we are given $f : Y \to X$, which we do not assume to be a toric morphism. By [Cox95a, Theorem 1.1] the data of such a map is equivalent to a Σ_X–collection on Y:

$$((L_\rho, u_\rho)_{\rho \in \Sigma_X(1)}, (\varphi_{m_x})_{m_x \in M_X})$$

In addition, [Cox95b] allows us to describe line bundles on Y and their global sections in terms of the homogeneous coordinates $(z_\tau)_{\tau \in \Sigma_Y(1)}$. All of these observations are combined into the following theorem, which is so useful that we will state it here in its entirety:

Theorem 2.6.1. [Cox95a, Theorem 3.2] The data of a morphism $f : Y \to X$ is the same as the data of homogeneous polynomials

$$P_\rho \in S^Y_{\beta_\rho}$$

for $\rho = f^*\mathcal{O}_X(D_\rho) \in \Sigma_X(1)$, where $\beta_\rho \in \operatorname{Pic} Y$ and $S^Y_{\beta_\rho}$ is the corresponding graded piece of the Cox ring:

$$S^Y = \mathbb{C}[z_\tau : \tau \in \Sigma_Y(1)]$$

These data are required to satisfy the following two conditions:

(1) $\sum_{\rho \in \Sigma_X(1)} \beta_\rho \otimes n_\rho = 0$ in $\operatorname{Pic} Y \otimes N_X$, where n_ρ is the principal generator of the ray ρ.
(2) $(P_\rho(z_\tau)) \notin Z(\Sigma_X) \subseteq \mathbb{A}^{\Sigma_X(1)}$ whenever $(z_\tau) \notin Z(\Sigma_Y) \subseteq \mathbb{A}^{\Sigma_Y(1)}$.

Furthermore, two such sets of data (P_ρ) and (P'_ρ) correspond to the same morphism if and only if there exists a $\lambda \in \operatorname{Hom}_\mathbb{Z}(\operatorname{Pic} X, \mathbb{G}_\mathrm{m})$ such that

$$\lambda(D_\rho) \cdot P_\rho = P'_\rho$$

for all $\rho \in \Sigma_X(1)$. Finally, if we define $\tilde{f}(z_\tau) = (P_\rho(z_\tau))$ then this defines a lift of f to the prequotients:

$$\begin{array}{ccc} \mathbb{A}^{\Sigma_Y(1)} \setminus Z(\Sigma_Y) & \xrightarrow{\tilde{f}} & \mathbb{A}^{\Sigma_X(1)} \setminus Z(\Sigma_X) \\ \big\downarrow{\scriptstyle q_Y} & & \big\downarrow{\scriptstyle q_X} \\ Y & \xrightarrow{\quad f \quad} & X \end{array}$$

Aside 2.6.1. Throughout this section we will stick to the notation established above; in particular we will use ρ to denote a ray in $\Sigma_X(1)$ and τ to denote a ray in $\Sigma_Y(1)$.

Recall our goal: given a map $f\colon Y \to X$ we wish to define a "push-forward" map:

$$\mathcal{Q}(f) : \mathcal{Q}_{g,n}(Y,\beta) \to \mathcal{Q}_{g,n}(X, f_*\beta)$$

Consider therefore a quasimap

$$((C, x_1, \ldots, x_n), (L_\tau, u_\tau)_{\tau \in \Sigma_Y(1)}, (\varphi_{m_Y})_{m_Y \in M_Y}) \in \mathcal{Q}_{g,n}(Y,\beta)$$

over an arbitrary base. Pick data $(P_\rho)_{\rho \in \Sigma_X(1)}$ corresponding to the map f, as in the theorem above; we will later see that our construction does not depend on this choice.

The idea of the construction is as follows. Locally around a point $x \in U_x \subseteq C$ we can trivialise the L_τ to obtain a morphism to the prequotient

$$(u_\tau)_\tau \colon U_x \to \mathbb{A}^{\Sigma_Y(1)}$$

which lifts the induced rational map to Y. On the other hand the data of $(P_\rho)_\rho$ gives a lifting of $f\colon Y \to X$ to a morphism between the prequotients

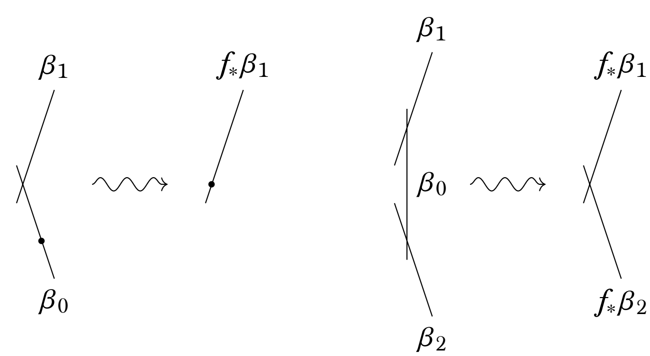
$$(P_\rho)_\rho \colon \mathbb{A}^{\Sigma_Y(1)} \to \mathbb{A}^{\Sigma_X(1)}$$

and so the composed map to the prequotient of X is given by:

$$(P_\rho((u_\tau)_\tau))_\rho : U_x \to \mathbb{A}^{\Sigma_X(1)}$$

In order to define the pushed-forward quasimap, we thus need to make sense of $P_\rho((u_\tau)_\tau)$ as a section of a certain line bundle $\tilde{L}_\rho$ on the curve.

We now make this precise. The first issue to address is the stabilisation of the source curve. The procedure is the same as in the case of stable maps: if $C_0 \subseteq C$ is a rational component with 2 special points (hence with curve class $\beta_0 > 0$) and such that $f_*\beta_0 = 0$, then C_0 should be contracted when we pass to X. The possibilities are:



To perform the contraction we need to construct a line bundle on C that is trivial on the components which we wish to contract and ample (relative to the base) on all the other components.

Fix a polarisation $\mathcal{O}_X(1)$ on X and express $f^*\mathcal{O}_X(1)$ in terms of the toric divisors of Y:

$$f^*\mathcal{O}_X(1) = \bigotimes_\tau \mathcal{O}_Y(D_\tau)^{\otimes c_\tau}$$

Then the line bundle

$$\omega_\pi(x_1 + \ldots + x_n) \otimes \bigotimes_\tau L_\tau^{\otimes a_\tau}$$

gives the required contraction by taking relative Proj. We thus obtain a curve $\tilde{C}$ and a morphism $\phi : C \to \tilde{C}$ which contracts the unstable components.

Recall that for $\rho \in \Sigma_X(1)$, P_ρ is a polynomial in the z_τ. We can thus write it as

$$P_\rho(z_\tau) = \sum_{\underline{a}} P_\rho^{\underline{a}}(z_\tau) = \sum_{\underline{a}} \mu_{\underline{a}} \prod_\tau z_\tau^{a_\tau} \tag{2.6.1}$$

where the sum is over a finite number of multindices $\underline{a} = (a_\tau) \in \mathbb{N}^{\Sigma_Y(1)}$ and the $\mu_{\underline{a}}$ are nonzero scalars. Observe that, for each $\underline{a}$, the line bundle $\otimes_\tau L_\tau^{\otimes a_\tau}$ on C is trivial on the components contracted by ϕ (which are always rational). Hence, by cohomology and base-change, it descends to a line bundle on $\tilde{C}$:

$$\tilde{L}_\rho^{\underline{a}} := \phi_* \bigotimes_\tau L_\tau^{\otimes a_\tau}$$

We may then take the following section of $\tilde{L}_\rho^{\underline{a}}$:

$$\tilde{u}_\rho^{\underline{a}} = P_\rho^{\underline{a}}(u_\tau) := \mu_{\underline{a}} \prod_\tau u_\tau^{a_\tau};$$

A priori this is really a section of $\otimes_\tau L_\tau^{\otimes a_\tau}$ on C; but since it is constant on the components contracted by ϕ, it descends to a section of $\tilde{L}_\rho^{\underline{a}}$ on $\tilde{C}$.

Thus, each of the terms $P_\rho^{\underline{a}}$ of P_ρ defines a section $\tilde{u}_\rho^{\underline{a}}$ of a line bundle $\tilde{L}_\rho^{\underline{a}}$ on $\tilde{C}$. What we want, however is a single section $\tilde{u}_\rho$ of a single line bundle $\tilde{L}_\rho$. This is where the isomorphisms φ_{m_Y} come in.

Recall that we have a short exact sequence:

$$0 \longrightarrow M_Y \xrightarrow{\theta} \mathbb{Z}^{\Sigma_Y(1)} \longrightarrow \operatorname{Pic} Y \longrightarrow 0 \tag{2.6.2}$$

Let $\underline{a}$ and $\underline{b}$ be multindices appearing in the sum (2.6.1) above. By the homogeneity of P_ρ we have

$$\sum_\tau a_\tau [D_\tau] = \beta_\rho = \sum_\tau b_\tau [D_\tau]$$

which is precisely the statement that in the above sequence $\underline{a}$ and $\underline{b}$ map to the same element of $\operatorname{Pic} Y$ (namely β_ρ). Hence there exists a unique $m_Y \in M_Y$ such that:

$$\theta(m_Y) = \underline{a} - \underline{b}$$

Now, the isomorphism φ_{m_Y} (contained in the data of our original quasimap) is a map:

$$\varphi_{m_Y} : \bigotimes_\tau L_\tau^{\otimes \langle m_Y, n_\tau \rangle} \cong \mathcal{O}_C$$

By definition, $\theta(m_Y) = (\langle m_Y, n_\tau \rangle)_{\tau \in \Sigma_Y(1)}$. But also $\theta(m_Y) = (a_\tau - b_\tau)_{\tau \in \Sigma_Y(1)}$. Hence we have:

$$\varphi_{m_Y} : \bigotimes_\tau L_\tau^{\otimes a_\tau} \cong \bigotimes_\tau L_\tau^{\otimes b_\tau}$$

This descends to give canonical isomorphisms

$$\tilde{L}_\rho^{\underline{a}} \cong \tilde{L}_\rho^{\underline{b}}$$

for all $\underline{a}$ and $\underline{b}$. Let us choose one such $\underline{a}$ (it doesn't matter which); call it $\underline{a}^\rho$. We define:

$$\tilde{L}_\rho := \tilde{L}_\rho^{\underline{a}^\rho}$$

Then for all $\underline{b}$ we can use the above isomorphisms to view $\tilde{u}_\rho^{\underline{b}}$ as a section of $\tilde{L}_\rho$. Summing all of these together we thus obtain a section $\tilde{u}_\rho$ of $\tilde{L}_\rho$, which we can write (with abuse of notation) as:

$$\tilde{u}_\rho = \sum_{\underline{a}} \mu_{\underline{a}} \prod_\tau u_\tau^{a_\tau}$$

Note that if we had made a different choice of $\underline{a}^\rho$ above the result would have been isomorphic.

Thus far we have constructed line bundles and sections $(\tilde{L}_\rho, \tilde{u}_\rho)_{\rho \in \Sigma_X(1)}$ on $\tilde{\mathcal{C}}$. It remains to define the isomorphisms

$$\tilde{\varphi}_{m_X} : \otimes_\rho \tilde{L}_\rho^{\otimes \langle m_X, n_\rho \rangle} \cong \mathcal{O}_{\tilde{C}}$$

for all $m_X \in M_X$. The left hand side is:

$$\otimes_\rho \tilde{L}_\rho^{\otimes \langle m_X, n_\rho \rangle} = \otimes_\rho \left(\phi_* \otimes_\tau L_\tau^{\otimes a_\tau^\rho} \right)^{\otimes \langle m_X, n_\rho \rangle} = \phi_* \otimes_\tau L_\tau^{\otimes \left(\sum_\rho a_\tau^\rho \langle m_X, n_\rho \rangle \right)}$$

Now, for $m_Y \in M_Y$ we have isomorphisms $\varphi_{m_Y} : \otimes_\tau L_\tau^{\otimes \langle m_Y, n_\tau \rangle} \cong \mathcal{O}_C$. In order to construct $\tilde{\varphi}_{m_X}$ it is therefore tempting to look for an m_Y such that

$$\langle m_Y, n_\tau \rangle = \sum_\rho a_\tau^\rho \langle m_X, n_\rho \rangle$$

for all $\tau \in \Sigma_Y(1)$ (we will then set $\tilde{\varphi}_{m_X} = \varphi_{m_Y}$). Consider therefore the short exact sequence (2.6.2). Recall that $\theta(m_Y) = (\langle m_Y, n_\tau \rangle)_{\tau \in \Sigma_Y(1)}$. Hence we need to show that

$$\left(\sum_\rho a_\tau^\rho \langle m_X, n_\rho \rangle \right)_{\tau \in \Sigma_Y(1)}$$

belongs to the image of θ, i.e. that it belongs to the kernel of the second map (notice that m_Y is then unique because θ is injective). This is equivalent to saying that

$$\sum_\tau \sum_\rho a_\tau^\rho \langle m_X, n_\rho \rangle [D_\tau] = 0 \in \operatorname{Pic} Y$$

Now, we have

$$\sum_\tau a_\tau^\rho [D_\tau] = \beta_\rho$$

so that the above sum becomes

$$\sum_\rho \langle m_X, n_\rho \rangle \beta_\rho = \left\langle m_X, \sum_\rho \beta_\rho \otimes n_\rho \right\rangle = \langle m_X, 0 \rangle = 0$$

where $\sum_\rho \beta_\rho \otimes n_\rho = 0$ by Condition (1) in Theorem 2.6.1. So there does indeed exist a (unique) $m_Y \in M_Y$ such that $\langle m_Y, n_\tau \rangle = \sum_\rho a_\tau^\rho \langle m_X, n_\rho \rangle$, and we can set:

$$\tilde{\varphi}_{m_X} = \varphi_{m_Y} : \bigotimes_\rho \tilde{L}_\rho^{\otimes \langle m_X, n_\rho \rangle} \cong \mathcal{O}_{\overline{C}}$$

Thus we have produced a quasimap with target X and class $f_* \beta$ on the base $\mathcal{Q}_{g,n}(Y, \beta)$:

$$((\tilde{C}, \tilde{x}_1, \ldots, \tilde{x}_n), (\tilde{L}_\rho, \tilde{u}_\rho)_{\rho \in \Sigma_X(1)}, (\tilde{\varphi}_{m_X})_{m_X \in M_X})$$

The proof that this construction does not depend on the choice of (P_ρ) is straightforward and is left to the reader.

It remains to demonstrate that the quasimap thus constructed is nondegenerate and stable. Nondegeneracy follows immediately from Condition (2) in Theorem 2.6.1. Put differently: the original quasimap defined a rational map $C \dashrightarrow Y$, whereas the new quasimap defines a rational map which is simply the composition $C \dashrightarrow Y \to X$ (up to contracting unstable components). Therefore the set of basepoints is exactly the same.

Stability follows precisely from the construction of ϕ: if we write the polarisation of X as $\mathcal{O}_X(1) = \bigotimes_\rho \mathcal{O}(D_\rho)^{\otimes b_\rho}$ then

$$\omega_{\bar{\pi}}(\bar{x}_1 + \ldots \bar{x}_n) \otimes \bigotimes_\rho \tilde{L}_\rho^{\otimes b_\rho}$$

will be $\tilde{\pi}$-ample on $\tilde{C}$, since we have contracted all the components on which $f^*\mathcal{O}_X(1)$ was trivial without introducing any rational tail.

To summarise, we have explained how to canonically associate, to any family of quasimaps with target Y, a family of quasimaps with target X. This completes the proof of the following:

Theorem 2.6.2. Let X and Y be smooth proper toric varieties and $f : Y \to X$ a morphism. Then there exists a natural push-forward map:

$$\mathcal{Q}(f) \colon \mathcal{Q}_{g,n}(Y, \beta) \to \mathcal{Q}_{g,n}(X, f_*\beta)$$

Remark 2.6.2. Theorem 2.6.1 tells us that we can lift any morphism between toric varieties to an equivariant morphism between the prequotients

$$\begin{array}{ccc} \mathbb{A}^{\Sigma_Y(1)} \setminus Z(\Sigma_Y) & \xrightarrow{\tilde{f}} & \mathbb{A}^{\Sigma_X(1)} \setminus Z(\Sigma_X) \\ \downarrow{\scriptstyle q_Y} & & \downarrow{\scriptstyle q_X} \\ Y & \xrightarrow{f} & X \end{array}$$

where the torus homomorphism

$$G_Y = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(Y), \mathbb{G}_{\mathrm{m}}) \to G_X = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(Y), \mathbb{G}_{\mathrm{m}})$$

is induced in the obvious way by $f \colon Y \to X$. Now, thinking of quasimaps as maps to the quotient stack, functoriality is again clear by postcomposition with $\tilde{f}$ (notice that the preimage of the unstable locus of X is the unstable locus of Y).

Finally, let us describe how this push-forward morphism behaves when f is a nonconstant map $\mathbb{P}^r \to \mathbb{P}^N$. Write f in homogeneous coordinates as:

$$f[z_0, \ldots, z_r] = [f_0(z_0, \ldots, z_r), \ldots, f_N(z_0, \ldots, z_r)]$$

where the f_i are all homogeneous of degree $a > 0$. Then given a quasimap with target $\mathbb{P}^r$

$$(C, L, u_o, \ldots, u_r)$$

the pushed-forward quasimap with target $\mathbb{P}^N$ is:

$$(C, L^{\otimes a}, f_0(u_0, \ldots, u_r), \ldots, f_N(u_0, \ldots, u_r))$$

2.6.2. **Relative obstruction theories for $\mathcal{Q}(Y) \to \mathcal{Q}(X)$.** Assume now that $f\colon Y \to X$ is a morphism (between projective varieties) satisfying any of the following three equivalent conditions:

(1) f is finite;
(2) for any ample line bundle $\mathcal{O}_X(1)$ on X, $f^*\mathcal{O}_X(1)$ is ample on Y;
(3) for every effective curve class $\beta \in \mathrm{H}_2^+(Y)$, $f_*\beta \neq 0$.

These conditions are in particular satisfied when f is a closed embedding, which is the case of most interest to us.

Observe then that the induced morphism

$$k = \mathcal{Q}(f)\colon \mathcal{Q}_{g,n}(Y,\beta) \to \mathcal{Q}_{g,n}(X, f_*\beta)$$

commutes with the projections to $\mathfrak{M}_{g,n}$, i.e. there is no need to stabilise the underlying curve. We would like to have a pull-back morphism $k^!$ between Chow groups. However, even in the easiest possible case when $f\colon Y \hookrightarrow X$ is a regular embedding, k itself is not necessarily a regular embedding, and so the Gysin map in the sense of [Ful98] is not guaranteed to exist.

However, when $\mathcal{Q}_{g,n}(X, f_*\beta)$ is unobstructed (for instance when $X = \mathbb{P}^N$ and $g = 0$ or $(g,n) = (1,0)$) there is a way around this. In [Man12] a generalisation of the Gysin map called the *virtual pull-back* is defined for morphisms endowed with a relative perfect obstruction theory. Moreover, a sufficient condition is given [Man12, Corollary 4.9] for this map to respect the virtual classes.

Lemma 2.6.3. For a *finite* morphism of smooth toric varieties $f\colon Y \to X$, there exists a relative obstruction theory $\mathbf{E}_k$ for the morphism

$$k : \mathcal{Q}_{g,n}(Y,\beta) \to \mathcal{Q}_{g,n}(X, f_*\beta)$$

which fits into a compatible triple with the standard obstruction theories for the quasimap spaces over $\mathfrak{M}_{g,n}$. Furthermore, $\mathbf{E}_k$ is perfect *if $\mathcal{Q}_{g,n}(X, f_*\beta)$ is unobstructed*, so that:

$$k_{\mathrm{v}}^![\mathcal{Q}_{g,n}(X, f_*\beta)] = [\mathcal{Q}_{g,n}(Y,\beta)]^{\mathrm{virt}}$$

Proof. Note first that, since k does not change the source curve of a quasimap, we indeed have a commuting triangle:

$$\begin{array}{ccc} \mathcal{Q}_{g,n}(Y,\beta) & \xrightarrow{\quad k \quad} & \mathcal{Q}_{g,n}(X,f_*\beta) \\ & \searrow \quad \swarrow & \\ & \mathfrak{M}_{g,n} & \end{array}$$

We have perfect obstruction theories $\mathbf{E}_{\mathcal{Q}(Y)/\mathfrak{M}}$ and $\mathbf{E}_{\mathcal{Q}(X)/\mathfrak{M}}$ and we want to find a perfect obstruction theory $\mathbf{E}_k$. Consider the diagram of universal curves

$$\begin{array}{ccc} \mathcal{C}_Y & \xrightarrow{\quad \alpha \quad} & \mathcal{C}_X \\ \downarrow \pi & \square & \downarrow \rho \\ \mathcal{Q}_{g,n}(Y,\beta) & \xrightarrow{k} & \mathcal{Q}_{g,n}(X,f_*\beta) \end{array}$$

which is cartesian because k does not alter the source curve of any quasimap. We have sheaves $\mathcal{F}_Y$ and $\mathcal{F}_X$ on $\mathcal{C}_Y$ and $\mathcal{C}_X$ respectively such that:

$$\mathbf{E}^\vee_{\mathcal{Q}(Y)/\mathfrak{M}} = \mathbf{R}^\bullet \pi_* \mathcal{F}_Y$$

$$\mathbf{E}^\vee_{\mathcal{Q}(X)/\mathfrak{M}} = \mathbf{R}^\bullet \rho_* \mathcal{F}_X$$

It follows (by flatness of ρ) that when we pull back the latter obstruction theory to $\mathcal{Q}(Y)$ we obtain:

$$k^*\mathbf{E}^\vee_{\mathcal{Q}(X)/\mathfrak{M}} = \mathbf{R}^\bullet \pi_* \alpha^* \mathcal{F}_X$$

To construct a compatible triple, we require a morphism $k^*\mathbf{E}_{\mathcal{Q}(X)/\mathfrak{M}} \to \mathbf{E}_{\mathcal{Q}(Y)/\mathfrak{M}}$. Dually, it is therefore enough to construct a morphism of sheaves on $\mathcal{C}_Y$

$$\mathcal{F}_Y \to \alpha^* \mathcal{F}_X$$

and then apply $\mathbf{R}^\bullet \pi_*$. This is analogous to the morphism $u^*T_Y \to u^*f^*T_X|_Y$ which is used in the stable maps setting. However the construction for quasimaps requires a little more ingenuity, because we do not have access to a universal map f.

The sheaf $\mathcal{F}_Y$ is defined on $\mathcal{C}_Y$ by the short exact sequence

$$0 \to \mathcal{O}_{\mathcal{C}_Y}^{\oplus r_Y} \to \oplus_\tau \mathcal{L}_\tau \to \mathcal{F}_Y \to 0$$

where $r_Y = \operatorname{rk}\operatorname{Pic} Y$ (implicitly we have chosen a basis for $\operatorname{Pic} Y$). Similarly $\mathcal{F}_X$ is defined on $\mathcal{C}_X$ by:

$$0 \to \mathcal{O}_{\mathcal{C}_X}^{\oplus r_X} \to \oplus_\rho \mathcal{L}_\rho \to \mathcal{F}_X \to 0$$

We will construct the map $\mathcal{F}_Y \to \alpha^*\mathcal{F}_X$ by first constructing a morphism:

$$\oplus_\tau \mathcal{L}_\tau \to \alpha^*(\oplus_\rho \mathcal{L}_\rho)$$

Recall that $f\colon Y \to X$ is given by homogeneous polynomials

$$P_\rho \in S^Y_{\beta_\rho} \subset S^Y = \mathbb{C}[z_\tau : \tau \in \Sigma_Y(1)]$$

in the Cox ring of Y, where $\beta_\rho = f^*[D_\rho] \in \operatorname{Pic} Y$. For all monomials appearing in P_ρ, if we look at their exponents $(a_\tau)_{\tau\in\Sigma_Y(1)}$, we have $\sum_{\tau\in\Sigma_Y(1)} a_\tau[D_\tau] = \beta_\rho$ by homogeneity; hence we can use the isomorphisms parametrised by M_Y as in the proof of Proposition 2.6.2 above in order to interpret the (P_ρ) as a morphism

$$(P_\rho)_{\rho\in\Sigma_X(1)}\colon \bigoplus_\tau \mathcal{L}_\tau \to \bigoplus_\rho \bigotimes_\tau \mathcal{L}_\tau^{\otimes a_\tau^\rho} = \bigoplus_\rho \tilde{\mathcal{L}}_\rho = \alpha^*\left(\bigoplus_\rho \mathcal{L}_\rho\right)$$

where the notation is as in the proof of functoriality in §2.6.1. Thus we have constructed a morphism $\oplus_\tau\mathcal{L}_\tau \to \alpha^*(\oplus_\rho\mathcal{L}_\rho)$.

On the other hand, $f\colon Y \to X$ induces a pullback map on line bundles $\operatorname{Pic}(X) \to \operatorname{Pic}(Y)$. Since we have implicitly chosen bases for these $\mathbb{Z}$-modules, this gives rise to a matrix, whose transpose we denote by:

$$Q \in \operatorname{Mat}_{r_X\times r_Y}(\mathbb{Z})$$

It is now clear by the functoriality construction that the left-hand square in the following diagram is commutative; hence it induces the (dashed) map of sheaves that we were after:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{C_Y}^{\oplus r_Y} & \longrightarrow & \oplus_\tau\mathcal{L}_\tau & \longrightarrow & \mathcal{F}_Y & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Q} & & \downarrow{\scriptstyle (P_\rho)} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{C_Y}^{\oplus r_X} & \longrightarrow & \alpha^*\left(\oplus_\rho\mathcal{L}_\rho\right) & \longrightarrow & \alpha^*\mathcal{F}_X & \longrightarrow & 0 \end{array} \tag{2.6.3}$$

Applying $\mathbf{R}^\bullet\pi_*$ and dualising we obtain a morphism between the obstruction theories for the quasimap spaces, and we can complete this to obtain an exact triangle

$$k^*\mathbf{E}_{\mathcal{Q}(X)/\mathfrak{M}} \to \mathbf{E}_{\mathcal{Q}(Y)/\mathfrak{M}} \to \mathbf{E}_k \xrightarrow{[1]}$$

on $\mathcal{Q}(Y)$. The axioms of a triangulated category then give a morphism of exact triangles:

$$\begin{array}{ccccccc} k^*\mathbf{E}_{\mathcal{Q}(X)/\mathfrak{M}} & \longrightarrow & \mathbf{E}_{\mathcal{Q}(Y)/\mathfrak{M}} & \longrightarrow & \mathbf{E}_k & \xrightarrow{[1]} & \\ \downarrow & & \downarrow & & \downarrow & & \\ k^*\mathbf{L}_{\mathcal{Q}(X)/\mathfrak{M}} & \longrightarrow & \mathbf{L}_{\mathcal{Q}(Y)/\mathfrak{M}} & \longrightarrow & \mathbf{L}_k & \xrightarrow{[1]} & \end{array}$$

It follows from a simple diagram chase that $\mathbf{E}_k \to \mathbf{L}_k$ is a relative obstruction theory supported in $[-2,0]$. On the other hand, assuming that $\mathcal{Q}_{g,n}(X, f_*\beta)$ is unobstructed, we may look at the long exact sequence in cohomology and find:

$$0 \to \mathrm{H}^{-2}(\mathbf{E}_k) \to \mathrm{H}^{-1}(k^*\mathbf{E}_{\mathcal{Q}(X)/\mathfrak{M}}) = 0$$

Hence $\mathrm{H}^{-2}(\mathbf{E}_k) = 0$ and so $\mathbf{E}_k$ is perfect. □

Remark 2.6.4. The short exact sequence defining $\mathcal{F}_X$ should be thought of as the pull-back of the Euler sequence

$$0 \to \mathcal{O}_X^{\oplus r_X} \to \bigoplus_{\rho \in \Sigma_X(1)} \mathcal{O}_X(D_\rho) \to \mathrm{T}_X \to 0$$

along the map $C \to X$, if such a map existed. In particular, if we work away from the locus of basepoints then $\mathcal{F}_X = u^*\,\mathrm{T}_X$.

In particular, for every smooth projective variety $i \colon X \hookrightarrow \mathbb{P}^N$, we have a virtual pull-back morphism

$$k_{\mathrm{v}}^! : \mathrm{A}_*(\mathcal{Q}_{0,n}(\mathbb{P}^N, d)) \to \mathrm{A}_*(\mathcal{Q}_{0,n}(X, \beta))$$

where $d = i_*\beta$, and more generally for any cartesian diagram

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & \square & \downarrow \\ \mathcal{Q}_{0,n}(X,\beta) & \xrightarrow{k} & \mathcal{Q}_{0,n}(\mathbb{P}^N, d) \end{array}$$

we get an associated virtual pull-back morphism:

$$k_{\mathrm{v}}^! : \mathrm{A}_*(G) \to \mathrm{A}_*(F)$$

This is used in §2.4 to pull-back the recursion formula for the pair $(\mathbb{P}^N, H)$ and obtain a recursion formula in the general case.

2.6.3. **Splitting axiom**. In this section we consider certain boundary strata of the moduli space of quasimaps, called *centipede loci*. These are the analogues in the absolute setting of the comb loci which appear in the relative setting (§2.3.2). The general element of such a locus has a source curve with $r+1$ irreducible components, one "trunk" of the centipede and r "legs." Each of these components has a prescribed genus, curve class and set of marked points.

Given such a locus, there are two natural virtual classes with which it can be equipped. One is the product virtual class induced by the absolute product of the $r+1$ quasimap spaces, and the other is the class pulled back from the ambient moduli space. In this section we show that these classes coincide. This is the quasimap version of the *splitting axiom* from Gromov–Witten theory, called the *cutting edges axiom* in [Beh99]. The fact that this extends to the quasimap setting has been discussed in [CFK17, §2.3.3]; here we spell out the details.

We first establish notation. Fix a smooth projective toric variety X and numerical invariants g, n, β such that the corresponding quasimap space is defined. Now fix partitions $G = (g_0, \ldots, g_r)$ of the genus, $A = (A_0, \ldots, A_r)$ of the marked points and $B = (\beta_0, \ldots, \beta_r)$ of the curve class and consider the following space (which we call the *centipede locus*):

$$\mathcal{D}^{\mathcal{Q}}(X, G, A, B) := \mathcal{Q}_{g_0, A_0 \cup \{q_1^0, \ldots, q_r^0\}}(X, \beta_0) \times_{X^r} \prod_{i=1}^{r} \mathcal{Q}_{g_i, A_i \cup \{q_i^1\}}(X, \beta_i)$$

Of course we assume that every element of the partition is in the stable range, so that every factor in the above product makes sense. See Remark 2.3.9 for a justification of why these are the correct boundary strata to consider. We can equip the centipede locus with the product virtual class in the following way. Set

$$\mathcal{E}^{\mathcal{Q}}(X, G, A, B) := \mathcal{Q}_{g_0, A_0 \cup \{q_1^0, \ldots, q_r^0\}}(X, \beta_0) \times \prod_{i=1}^{r} \mathcal{Q}_{g_i, A_i \cup \{q_i^1\}}(X, \beta_i)$$

which we endow with the product class:

$$[\mathcal{E}^{\mathcal{Q}}(X, G, A, B)]^{\text{virt}} := [\mathcal{Q}_{g_0, A_0 \cup \{q_1^0, \ldots, q_r^0\}}(X, \beta_0)]^{\text{virt}} \times \prod_{i=1}^{r} [\mathcal{Q}_{g_i, A_i \cup \{q_i^1\}}(X, \beta_i)]^{\text{virt}}$$

We then consider the cartesian diagram:

$$\begin{array}{ccc} \mathcal{D}^{\mathcal{Q}}(X, G, A, B) & \xrightarrow{h} & \mathcal{E}^{\mathcal{Q}}(X, G, A, B) \\ \downarrow \mathrm{ev}_q & \square & \downarrow \mathrm{ev}_q \\ X^r & \xrightarrow{\Delta_{X^r}} & X^r \times X^r \end{array} \tag{2.6.4}$$

Since X is smooth Δ_{X^r} is a regular embedding, so we have a Gysin map which we use to define:

$$[\mathcal{D}^{\mathcal{Q}}(X, G, A, B)]^{\text{virt}} := \Delta_{X^r}^{!} [\mathcal{E}^{\mathcal{Q}}(X, G, A, B)]^{\text{virt}}$$

Notice that if we set

$$\mathfrak{M}^{\text{wt}}_{G,A,B} := \mathfrak{M}^{\text{wt}}_{g_0, A_0 \cup \{q_1^0, \ldots, q_r^0\}, \beta_0} \times \prod_{i=1}^{r} \mathfrak{M}^{\text{wt}}_{g_i, A_i \cup \{q_i^1\}, \beta_i}$$

then there is a morphism given by forgetting everything except the source curves and their classes

$$\rho_E : \mathcal{E}^{\mathcal{Q}}(X, G, A, B) \to \mathfrak{M}^{\text{wt}}_{G,A,B}$$

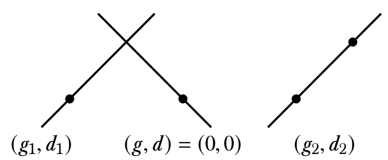
and the virtual class on $\mathcal{E}^{\mathcal{Q}}(X, G, A, B)$ is induced by a perfect obstruction theory $\mathbf{E}_{\rho_E} \to \mathbf{L}_{\rho_E}$ given by the product of the standard obstruction theories for each factor:

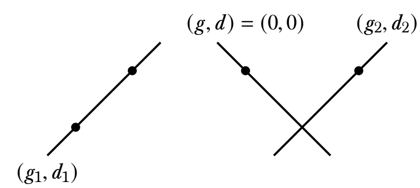
$$\mathcal{Q}_{g_i, A_i \cup \{q_i\}}(X, \beta_i) \to \mathfrak{M}^{\mathrm{wt}}_{g_i, A_i, \beta_i}$$

On the other hand, we have the following cartesian diagram

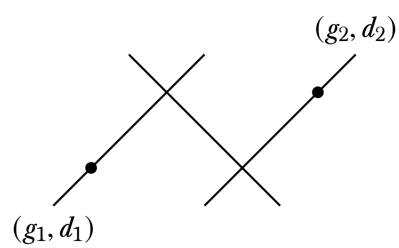
$$\begin{array}{ccc} \mathcal{D}^{\mathcal{Q}}(X, G, A, B) & \xrightarrow{\varphi} & \mathcal{Q}_{g,n}(X, \beta) \\ \downarrow \rho_D & \square & \downarrow \rho_Q \\ \mathfrak{M}^{\mathrm{wt}}_{G,A,B} & \xrightarrow{\psi} & \mathfrak{M}^{\mathrm{wt}}_{g,n,\beta} \end{array} \tag{2.6.5}$$

The bottom horizontal map is not a closed embedding: due to the existence of degree zero rational components, there may be many possible equally valid ways of breaking up a nodal curve. For instance, the following two elements of $\mathfrak{M}^{\mathrm{wt}}_{G,A,B}$





map to the same weighted curve under ψ, namely:



Nevertheless ψ has a natural perfect obstruction theory, given by $\mathbf{L}_\psi$: we only need to show that it is supported in $[-1, 0]$. Consider the exact triangle:

$$\psi^* \mathbf{L}_{\mathfrak{M}^{\mathrm{wt}}_{g,n,\beta}} \to \mathbf{L}_{\mathfrak{M}^{\mathrm{wt}}_{G,A,B}} \to \mathbf{L}_\psi \xrightarrow{[1]}$$

The first two terms are concentrated in degrees $[0, 1]$, because they are the cotangent complexes of smooth Artin stacks. Therefore $\mathbf{L}_\psi$ is concentrated in degrees $[-1, 1]$. Furthermore, if we examine the long exact cohomology sequence near $\mathrm{H}^1(\mathbf{L}_\psi)$ we find

$$\mathrm{H}^1(\psi^* \mathbf{L}_{\mathfrak{M}^{\mathrm{wt}}_{g,n,\beta}}) \to \mathrm{H}^1(\mathbf{L}_{\mathfrak{M}^{\mathrm{wt}}_{G,A,B}}) \to \mathrm{H}^1(\mathbf{L}_\psi) \to 0$$

and hence we must show that the first map is surjective. But this is dual to the map which takes an infinitesimal automorphism of the disconnected curve to an infinitesimal automorphism of the corresponding connected curve (obtained by glueing together the "nodal" marked points). The space of infinitesimal automorphisms of a nodal curve splits into a direct sum of infinitesimal automorphisms of each component; since the glueing does not affect the components, we see that this map is an isomorphism. Hence $\mathrm{H}^1(\mathbf{L}_\psi) = 0$ as claimed; morally this follows from the fact that the fibres of ψ are Deligne–Mumford.

Hence there is a virtual pull-back map $\psi^!$ which defines a class

$$\psi^![\mathcal{Q}_{g,n}(X,\beta)]^{\mathrm{virt}}$$

on $\mathcal{D}^{\mathcal{Q}}(X,G,A,B)$. This is the same class as the one induced by the following perfect obstruction theory

$$\varphi^*\mathbf{E}_{\rho_Q} \to \mathbf{L}_{\rho_D}$$

by functoriality of virtual pull-backs.

Finally if we look at (2.6.4) we see that $\mathrm{ev}_q^* \mathbf{L}_{\Delta_{X^r}} \to \mathbf{L}_h$ is a perfect obstruction theory for the map h. To summarise, we have a triangle

$$\begin{array}{ccc} \mathcal{D}^{\mathcal{Q}}(X,G,A,B) & \xrightarrow{\quad h \quad} & \mathcal{E}^{\mathcal{Q}}(X,G,A,B) \\ & {\scriptstyle\rho_D}\searrow \quad \swarrow{\scriptstyle\rho_E} & \\ & \mathfrak{M}^{\mathrm{wt}}_{G,A,B} & \end{array} \tag{2.6.6}$$

where all three morphisms are equipped with perfect obstruction theories. We simply need to check that these fit together in a compatible triple.

Lemma 2.6.5. There is a compatible triple

$$(h^*\mathbf{E}_{\rho_E}, \varphi^*\mathbf{E}_{\rho_Q}, \mathrm{ev}_q^* \mathbf{L}_{\Delta_{X^r}})$$

for the triangle (2.6.6). Hence by functoriality of virtual pull-backs we have:

$$\psi^![\mathcal{Q}_{g,n}(X,\beta)]^{\mathrm{virt}} = \Delta^!_{X^r}[\mathcal{E}^{\mathcal{Q}}(X,G,A,B)]^{\mathrm{virt}} = [\mathcal{D}^{\mathcal{Q}}(X,G,A,B)]^{\mathrm{virt}}$$

Proof. We need to construct a morphism of triangles

$$\begin{array}{ccccccc} h^*\mathbf{E}_{\rho_E} & \longrightarrow & \varphi^*\mathbf{E}_{\rho_Q} & \longrightarrow & \mathrm{ev}_q^* \mathbf{L}_{\Delta_{X^r}} & \xrightarrow{[1]} & \\ \downarrow & & \downarrow & & \downarrow & & \\ h^*\mathbf{L}_{\rho_E} & \longrightarrow & \mathbf{L}_{\rho_D} & \longrightarrow & \mathbf{L}_h & \xrightarrow{[1]} & \end{array}$$

Consider the following diagram:

$$\begin{array}{ccccc} h^*\tilde{C} & \xrightarrow{\nu} & \varphi^*\mathcal{C} & \longrightarrow & \mathcal{C} \\ & \searrow^{\eta} & \downarrow & \square & \downarrow \pi \\ & & \mathcal{D}^{\mathcal{Q}}(X,G,A,B) & \xrightarrow{\varphi} & \mathcal{Q}_{g,n}(X,\beta) \end{array}$$

Here $\tilde{C}$ is the universal (disconnected) curve over $\mathcal{E}^{\mathcal{Q}}(X,G,A,B)$, which we have pulled back to $\mathcal{D}^{\mathcal{Q}}(X,G,A,B)$, while $\varphi^*\mathcal{C}$ is the universal curve over $\mathcal{D}^{\mathcal{Q}}(X,G,A,B)$. Therefore the map $\nu : h^*\tilde{\mathcal{C}} \to \varphi^*\mathcal{C}$ is (fiberwise) a partial normalisation map given by normalising the nodes which connect the "trunk" of the centipede to the "legs."

There are natural sheaves $\mathcal{F}$ and $\tilde{\mathcal{F}}$ on $\mathcal{C}$ and $h^*\tilde{\mathcal{C}}$ respectively, such that

$$\varphi^*\mathbf{E}^{\vee}_{\rho_Q} = \mathbf{R}^{\bullet}\pi_*\mathcal{F}$$
$$h^*\mathbf{E}^{\vee}_{\rho_E} = \mathbf{R}^{\bullet}\eta_*\tilde{\mathcal{F}}$$

Furthermore $\nu^*\mathcal{F} \simeq \tilde{\mathcal{F}}$, hence by tensoring the partial normalisation short exact sequence

$$0 \to \mathcal{O}_{\varphi^*\mathcal{C}} \to \nu_*\mathcal{O}_{h^*\tilde{\mathcal{C}}} \to \mathcal{O}_q \to 0$$

with $\mathcal{F}$ and applying the projection formula, we obtain

$$0 \to \mathcal{F} \to \nu_*\tilde{\mathcal{F}} \to \mathcal{F}_q \to 0$$

on $\varphi^*\mathcal{C}$, where q is the locus of nodes connecting the trunk to the spine. (The fact that the morphism on the left is injective follows by applying the Snake Lemma to the short exact sequence defining $\mathcal{F}$.) To this we can apply $\mathbf{R}^{\bullet}\pi_*$ to obtain an exact triangle

$$\mathbf{R}^{\bullet}\pi_*\mathcal{F} \to \mathbf{R}^{\bullet}\eta_*\tilde{\mathcal{F}} \to \mathbf{R}^{\bullet}\pi_*\mathcal{F}_q \xrightarrow{[1]} \tag{2.6.7}$$

Finally, notice that, since quasimaps are required not to have base-points at the nodes, the fibre of the sheaf $\mathcal{F}$ at each of the nodes q can actually be identified with the tangent to the toric variety X at the image of the node itself, i.e. $\mathbf{R}^{\bullet}\pi_*\mathcal{F}_q \cong \mathrm{ev}_q^*\,\mathrm{T}_{X^r} = \mathrm{ev}_q^*\,\mathrm{T}_{\Delta_{X^r}}[-1]$. Dualising sequence (2.6.7) we obtain

$$h^*\mathbf{E}_{\rho_E} \to \varphi^*\mathbf{E}_{\rho_Q} \to \mathrm{ev}_q^*\,\mathbf{E}_{\Delta_{X^r}} \xrightarrow{[1]}$$

as required. $\square$

2.6.4. **Comparison with the GIT construction.** Let X be a smooth projective toric variety and $Y \hookrightarrow X$ a smooth very ample hypersurface. The complete linear system $|\mathcal{O}_X(Y)|$ gives an embedding $i : X \hookrightarrow \mathbb{P}^N$ which expresses Y as the intersection inside $\mathbb{P}^N$ of X and a certain hyperplane H: $Y = X \cap H = i^{-1}(H)$. We can *define* the moduli space of quasimaps to Y via the following cartesian diagram:

$$\begin{array}{ccc} \mathcal{Q}_{g,n}(Y,\beta) & \longrightarrow & \mathcal{Q}_{g,n}(H,d) \\ \big\downarrow & \square & \big\downarrow \\ \mathcal{Q}_{g,n}(X,\beta) & \xrightarrow{k} & \mathcal{Q}_{g,n}(\mathbb{P}^N,d) \end{array}$$

where $d = i_*\beta$. This moduli space is easy to describe: let s_Y denote the section of $\mathcal{O}_X(Y)$ cutting out Y inside X. Recall from §2.2.3 that for any quasimap

$$((C, x_1, \ldots, x_n), (L_\rho, u_\rho)_{\rho \in \Sigma_X(1)}, (\varphi_m)_{m \in M_X}) \in \mathcal{Q}_{g,n}(X,\beta)$$

we can construct a section u_Y of a line bundle L_Y on C, which plays the role of the pull-back of s_Y to C. Then

$$\mathcal{Q}_{g,n}(Y,\beta) \subseteq \mathcal{Q}_{g,n}(X,\beta)$$

consists of those quasimaps such that $u_Y \equiv 0$.

The cartesian diagram above can also be used to endow $\mathcal{Q}_{g,n}(Y,\beta)$ with a virtual class via virtual (or diagonal) pull-back along k. Thus we can define quasimap invariants for Y.

On the other hand, Y has the natural structure of a GIT quotient

$$Y = C(Y) /\!\!/ G$$

where $C(Y) \subseteq \mathbb{A}^{\Sigma_X(1)}$ is the affine cone over Y and $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(X), \mathbb{G}_{\mathrm{m}}) \cong \mathbb{G}_{\mathrm{m}}^{r_X}$ acts on $C(Y)$ via the natural inclusion

$$\mathbb{G}_{\mathrm{m}}^{r_X} \hookrightarrow \mathbb{G}_{\mathrm{m}}^{\Sigma_X(1)}$$

(here $C(Y) \subseteq \mathbb{A}^{\Sigma_X(1)}$ is preserved by G because it is cut out by a homogeneous equation). In [CFKM14] moduli spaces of quasimaps are constructed for GIT quotient targets (satisfying a number of conditions, all of which hold for Y). There is thus a moduli space

$$\mathcal{Q}^{\mathrm{GIT}}_{g,n}(Y,\beta)$$

which admits a virtual class. Hence we have two moduli spaces of quasimaps to Y, each equipped with a virtual class, and we want to check that these definitions agree.

Objects of $\mathcal{Q}^{\mathrm{GIT}}_{g,n}(Y,\beta)$ are diagrams of the form

$$\begin{array}{ccc} P & \longrightarrow & C(Y) \\ \big\downarrow{\scriptstyle G} & & \\ C & & \end{array}$$

where C is a prestable curve, P is a principal G-bundle and the map $P \to C(Y)$ is G-equivariant. Equivalently, an object consists of a prestable curve C, a principal G-bundle P and a section u of the associated $C(Y)$-bundle:

$$\begin{array}{c} P \times_G C(Y) \\ {\scriptstyle p}\big\downarrow \big\uparrow{\scriptstyle u} \\ C \end{array}$$

The obstruction theory on this space is defined relative to the stack $\mathfrak{Bun}_G^{g,n}$ parametrising principal G-bundles on the universal curve:

$$\mathcal{C}_{\mathfrak{M}_{g,n}} \to \mathfrak{M}_{g,n}$$

It is given by

$$\mathbf{E}^\vee_{\mathcal{Q}/\mathfrak{Bun}_G} = \mathbf{R}^\bullet \pi_*(u^* \mathrm{T}_p)$$

where π is the universal curve over $\mathcal{Q} = \mathcal{Q}^{\mathrm{GIT}}_{g,n}(Y, \beta)$ and T_p is the relative tangent complex. There is a natural isomorphism

$$\mathfrak{Bun}_G^{g,n} \cong \times^{r_X}_{\mathfrak{M}_{g,n}} \mathfrak{Pic}_{\mathfrak{g},\mathfrak{n}}$$

given by sending P to the r_X individual factors of the affine bundle $P \times_G \mathbb{A}^{r_X}$. Furthermore there is a G-equivariant embedding

$$\begin{array}{ccc} P \times_G C(Y) & \overset{j}{\hookrightarrow} & P \times_G \mathbb{A}^{\Sigma_X(1)} \cong \bigoplus_{\rho \in \Sigma_X(1)} L_\rho \\ {\scriptstyle p}\big\downarrow \big\uparrow{\scriptstyle u} & \swarrow & \\ C & & \end{array}$$

which expresses $P \times_G C(Y)$ as the vanishing locus of u_Y in $\oplus_{\rho \in \Sigma_X(1)} L_\rho$. This shows that the two definitions of the moduli space agree.

Finally we must compare the virtual classes. Using the normal sheaf sequence for the inclusion j (relative to the base C) we obtain a short exact sequence on C:

$$0 \to u^* \mathrm{T}_p \to \bigoplus_{\rho \in \Sigma_X(1)} L_\rho \to u^* \mathrm{N}_{P \times_G C(Y)/\oplus_{\rho \in \Sigma_X(1)} L_\rho} \to 0$$

Since $P \times_G C(Y)$ is defined by the vanishing of u_Y, we see that the final term is isomorphic to the line bundle L_Y discussed above. Thus as elements of the derived category

$$u^* \mathrm{T}_p = \left[\bigoplus_{\rho \in \Sigma_X(1)} L_\rho \to L_Y \right]$$

Applying $\mathbf{R}^\bullet \pi_*$ we obtain on the left hand side the obstruction theory for the GIT moduli space relative $\mathfrak{Bun}_G^{g,n}$. On the other hand, the first term on the right hand side is the obstruction theory for $\mathcal{Q}(X)$ relative the product of the Picard stacks (isomorphic to $\mathfrak{Bun}_G^{g,n}$ via the discussion above) whereas the second term is the relative obstruction theory for $\mathcal{Q}(Y)$ inside $\mathcal{Q}(X)$. Thus the virtual classes agree, as claimed.

Appendix 2.7: Intersection-theoretic lemmas

In this appendix we explicitly define the *diagonal pull-back* along a morphism whose target is unobstructed (used in [Gat02]) and verify that this agrees with the virtual pull-back of [Man12] when both are defined. We also check that it satisfies some expected compatibility properties.

Consider a morphism of DM stacks $f \colon Y \to X$ over a smooth base $\mathfrak{M}$, such that X is smooth over $\mathfrak{M}$ and Y carries a virtual class given by a perfect obstruction theory $\mathbf{E}_{Y/\mathfrak{M}}$. Then, for every Cartesian diagram

$$\begin{array}{ccc} G & \xrightarrow{g} & F \\ \downarrow{\scriptstyle q} & \square & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & X \end{array}$$

and every class $\alpha \in \mathrm{A}_*(F)$, we may define

$$f_\Delta^!(\alpha) = \Delta_X^!([Y]^{\mathrm{vir}} \times \alpha) \in \mathrm{A}_*(G)$$

which we call the *diagonal pull-back*. We first show that it coincides with the usual virtual pull-back along f in the presence of a compatible perfect obstruction theory for f.

Lemma 2.7.1. Assume that there exists a relative obstruction theory $\mathbf{E}_f$ compatible with $\mathbf{E}_{Y/\mathfrak{M}}$ and the standard (unobstructed) obstruction theory for X, i.e:

$$\begin{array}{ccccccc} f^*\mathbf{L}_{X/\mathfrak{M}} & \longrightarrow & \mathbf{E}_{Y/\mathfrak{M}} & \longrightarrow & \mathbf{E}_f & \xrightarrow{[1]} & \\ \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow & & \downarrow & & \\ f^*\mathbf{L}_{X/\mathfrak{M}} & \longrightarrow & \mathbf{L}_{Y/\mathfrak{M}} & \longrightarrow & \mathbf{L}_f & \xrightarrow{[1]} & \end{array}$$

Then for every Cartesian diagram and every class $\alpha \in \mathrm{A}_*(F)$ as above,

$$f_{\mathrm{v}}^!(\alpha) = f_\Delta^!(\alpha).$$

Proof. Consider the following cartesian diagram:

$$
\begin{array}{ccccc}
G & \xrightarrow{q\times g} & Y \times_{\mathfrak{M}} F & \xrightarrow{\mathrm{pr}_1} & Y \\
\downarrow{\scriptstyle g} & \square & \downarrow{\scriptstyle f\times \mathrm{Id}} & \square & \downarrow{\scriptstyle f} \\
F & \xrightarrow{p\times \mathrm{Id}} & X \times_{\mathfrak{M}} F & \xrightarrow{\mathrm{pr}_1} & X \\
\downarrow{\scriptstyle p} & \square & \downarrow{\scriptstyle \mathrm{Id}\times p} & & \\
X & \xrightarrow{\Delta_X} & X \times_{\mathfrak{M}} X & &
\end{array}
$$

Then, by commutativity of (virtual) pull-backs, we have

$$
\begin{aligned}
\Delta_X^!([Y]^{\mathrm{vir}} \times \alpha) &= \Delta^!((f_{\mathrm{v}}^![X]) \times \alpha) \\
&= \Delta_X^!(f_{\mathrm{v}}^!([X] \times \alpha)) \\
&= f_{\mathrm{v}}^!(\Delta_X^!([X] \times \alpha)) \\
&= f_{\mathrm{v}}^!(\alpha)
\end{aligned}
$$

as required. □

Secondly, we show that the diagonal pull-back behaves similarly to an ordinary virtual pull-back (e.g. commutes with other virtual pull-backs) even in the absence of a compatible perfect obstruction theory.

Lemma 2.7.2. The diagonal pull-back morphism as defined above commutes with ordinary Gysin maps and with virtual pull-backs.

Proof. First consider the case of ordinary Gysin maps. We must consider a cartesian diagram:

$$
\begin{array}{ccccc}
Y'' & \longrightarrow & X'' & \longrightarrow & S \\
\downarrow & \square & \downarrow & \square & \downarrow{\scriptstyle k} \\
Y' & \longrightarrow & X' & \longrightarrow & T \\
\downarrow & \square & \downarrow & & \\
Y & \xrightarrow{f} & X & &
\end{array}
$$

with k a regular embedding and $f\colon Y \to X$ as before. We need to show that for all $\alpha \in A_*(X')$:

$$
k^! f_{\Delta}^!(\alpha) = f_{\Delta}^! k^!(\alpha)
$$

We form the cartesian diagram

$$\begin{array}{ccccc}
Y'' & \longrightarrow & Y \times X'' & \longrightarrow & S \\
\big\downarrow & \square & \big\downarrow & \square & \big\downarrow{\scriptstyle k} \\
Y' & \longrightarrow & Y \times X' & \longrightarrow & T \\
\big\downarrow & \square & \big\downarrow & & \\
X & \xrightarrow{\Delta_X} & X \times X & &
\end{array}$$

and apply commutativity of usual Gysin morphisms. In the case where k is not a regular embedding but rather is equipped with a relative perfect obstruction theory, the same argument works with $k^!$ replaced by $k^!_{\mathrm{v}}$. $\square$

CHAPTER 3

Towards a recursive formula for log Gromov–Witten invariants of hyperplane arrangements

This represents work in progress, and will appear in the form of an article once complete.

Abstract: We describe work in progress towards obtaining a Gathmann-like recursion formula for log Gromov–Witten invariants in genus zero. Along the way, we introduce auxiliary moduli spaces of log stable maps and compare them to the ordinary moduli spaces, providing some insights into the geometry of the latter. We present several example applications of our proposed formula.

3.1. Introduction

3.1.1. **Relative Gromov–Witten theory**. The theory of relative Gromov–Witten invariants has had a profound impact on enumerative geometry over the past two decades. Besides its intrinsic interest, it has found applications to numerous other areas, including ordinary Gromov–Witten theory [MP06] [Gat03b], intersection theory on the moduli space of curves [GV05] [JPPZ17] and the study of open string invariants [LLLZ09].

We start with a brief overview of the subject. Given X a smooth projective variety and $Y \subseteq X$ a smooth hypersurface, the relative Gromov–Witten invariants of the pair (X, Y) are defined as (virtual) counts of stable maps to X with fixed orders of tangency to Y at the marked points.

To be more precise: fix a genus $g \geq 0$, a number of marked points $n \geq 0$, a curve class $\beta \in \mathrm{H}_2^+(X)$ and a vector of tangency orders $\alpha = (\alpha_1, \ldots, \alpha_n)$. Each α_i is a non-negative integer which will record the tangency order of the stable map f to Y at the marked point $x_i \in C$. Given such data, one wishes to construct a suitable moduli space

$$\overline{\mathcal{M}}_{g,\alpha}(X|Y, \beta)$$

of relative stable maps to (X, Y) with tangency orders α. This should be a proper Deligne–Mumford stack of finite type, with a virtual fundamental class of dimension equal to the virtual dimension:

$$\operatorname{vdim} \overline{\mathcal{M}}_{g,\alpha}(X|Y,\beta) = \operatorname{vdim} \overline{\mathcal{M}}_{g,n}(X,\beta) - \sum_{i=1}^{n} \alpha_i$$

It turns out that there are many different ways to define such a space [Gat02] [Li01, Li02] [AF16] [Kim10] [GS13, Che14, AC14], though the resulting invariants are always the same [AMW14]. The choice of which moduli space to work with largely depends on one's intended application, since each approach comes with its own particular advantages and limitations.

Here we will discuss the approach pursued by A. Gathmann in genus zero [Gat02], since it is the most relevant to our work. Fix a smooth projective variety X, a smooth very ample hypersurface $Y \subseteq X$, and the numerical data n, β, α described above. Then a stable map

$$(C, x_1, \ldots, x_n, f) \in \overline{\mathcal{M}}_{0,n}(X,\beta)$$

is said to be a *relative stable map* (of tangency α) if and only if the following two conditions are satisfied:

(1) $f(x_i) \in Y$ whenever $\alpha_i > 0$;
(2) the class $f^*[Y] - \sum_{i=1}^{n} \alpha_i x_i \in \mathrm{A}_0(f^{-1}(Y))$ is effective.

Notice that by Condition (1), each $\alpha_i x_i$ does indeed define a class in $f^{-1}(Y)$, so Condition (2) makes sense. This definition can be reformulated more explicitly as follows (see [Gat02, Remark 1.4]): a stable map is a relative stable map (of tangency α) if and only if, for every connected component Z of $f^{-1}(Y) \subseteq C$:

(1) if $Z = x_i$ consists of an isolated marked point, then the tangency order of f at x_i with respect to Y must be at least α_i (if Z consists of an isolated unmarked point then there is no condition);
(2) if Z is one-dimensional (hence a union of irreducible components of C), then if we let $C_1, \ldots, C_r$ denote the components of C adjacent to Z and $q_1, \ldots, q_r$ the corresponding connecting nodes, then we must have

$$Y \cdot f_*[Z] + \sum_{i=1}^{r} m_i \geq \sum_{x_i \in Z} \alpha_i$$

where m_i denotes the tangency order of $f|_{C_i}$ at q_i with respect to Y.

Obviously, in order for there to be any relative stable maps at all, the following inequality must be satisfied:

$$Y \cdot \beta \geq \sum_{i=1}^{n} \alpha_i$$

It is important to note that one does *not* require equality here (in contrast to other approaches to relative Gromov–Witten theory). As such, each α_i should be thought of only as a *lower bound* for the tangency at x_i.

These conditions define the moduli space of relative stable maps as a closed substack of the moduli space of absolute stable maps:

$$\overline{\mathcal{M}}^{\text{Gat}}_{0,\alpha}(X|Y,\beta) \subseteq \overline{\mathcal{M}}_{0,n}(X,\beta)$$

In particular, it is a proper Deligne–Mumford stack of finite type [Gat02, Definition 1.18]. It does *not* carry a natural perfect obstruction theory, but nevertheless it is possible to endow it with a virtual fundamental class with the desired properties (this is where the genus zero and very ample hypotheses are used). One can then define *relative Gromov–Witten invariants* in the usual way, by integrating this virtual class against evaluation and psi classes.

The obvious question which arises is: how do we compute these numbers? In [Gat02, §§2-3] Gathmann proves a formula, which expresses any relative invariant of (X,Y) in terms of absolute invariants of Y and relative invariants of (X,Y) with strictly lower tangency. This has been discussed in detail in §2.2.2.

Applying this formula recursively to the relative invariants of lower tangency which appear, we eventually remove all the tangencies and arrive at an expression involving only the absolute invariants of Y and X. (With some more work, it is actually possible to obtain an expression involving only the invariants of X; see [Gat02, Corollary 5.7].) This gives an effective algorithm for computing relative Gromov–Witten invariants. For low degree invariants it is possible to carry out this computation by hand; moreover, the algorithm is simple to implement on a computer (for the case of projective space, see [Gat]).

3.1.2. **Logarithmic Gromov–Witten theory**. So far, we have restricted ourselves to imposing tangencies along a single smooth hypersurface. But there are many interesting enumerative questions involving tangencies to multiple hypersurfaces.

From a Gromov–Witten theorist's perspective, what is needed in order to address such questions is a theory of relative Gromov–Witten invariants with respect to (certain) reducible divisors. The specific class of divisors we are interested in is the *snc divisors*; a divisor $D \subseteq X$ is snc if all of its components are smooth and their intersections locally look like the intersection of a collection of co-ordinate hyperplanes in affine space.

Given an snc divisor $D \subseteq X$, we want to be able to define relative Gromov–Witten invariants of (X, D). Logarithmic Gromov–Witten theory provides such a definition [GS13, Che14, AC14]. The basic idea is as follows: the snc divisor D defines a certain structure on X, called a log structure. A log stable map, roughly speaking, is a stable map $f\colon C \to X$ where C is also equipped with a log structure and f is enhanced to a morphism of log schemes. There is, of course, much more to be said on this subject; full details can be found in the above references.

Remark 3.1.1. Logarithmic Gromov–Witten invariants are defined more generally than discussed above; in fact, they make sense for arbitrary log smooth targets. In particular this allows one to define the Gromov–Witten invariants of a singular variety appearing as the central fibre of a toric degeneration. This demonstrates that log Gromov–Witten theory is the correct general context in which to discuss the degeneration formula [ACGS17] [KLR18]. Moreover, log Gromov–Witten theory has intimate connections to Mirror Symmetry via the Gross-Siebert program [GS16].

Despite the theory's tremendous importance, there have been relatively few explicit calculations of log Gromov–Witten invariants (notable exceptions include [MR16] and [ACGS17, §6]). Mostly this is due to the youth of the subject, but it also reflects the fact that the moduli spaces are somewhat difficult to describe explicitly, since moduli of log structures can be complicated in general.

3.1.3. **Outline**. In this chapter we present work in progress towards a Gathmann-style recursion formula for log Gromov–Witten invariants. To fix ideas and simplify some arguments, we work with $X = \mathbb{P}^N$ and D some subset of the toric boundary (though we believe our methods can be applied in greater generality than this).

We will describe two attempts at obtaining a recursion formula. The first, described in §§3.2–3.3, is via a study of an alternative system of moduli spaces, called the *snc moduli spaces of relative stable maps*. While ultimately this approach was not successful in producing a recursion formula, we believe it is still of interest, since the comparison results described in §3.3 yield a surprising amount of insight into the geometry of the moduli spaces of log stable maps.

The second attempt, described in §3.4, involves carrying out the recursion directly on the moduli space of log stable masp. This requires an understanding of the recursive structure of the boundary of this space, and in particular of how to glue log stable maps. As proof of concept, we provide a number of example computations. However, it should be clear that the full details of the recursion, and its proof, are still work in progress.

3.1.4. **Background**. We assume that the reader is somewhat familiar with Gathmann's moduli spaces of relative stable maps and his recursion formula [Gat02]; these are discussed in detail in §2.2.2. We also assume that the reader is familiar with the rudiments of logarithmic Gromov–Witten theory [GS13, Che14, AC14].

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3.2. First attempt: auxiliary moduli spaces

3.2.1. **Setup**. Throughout we will consider $X = \mathbb{P}^N$ and D some collection of co-ordinate hyperplanes, which without loss of generality we may take to be

$$D = \sum_{i=0}^{r} H_i$$

for some $0 \leq r \leq N$ (as mentioned before, we believe our results should apply in greater generality than this, but in the first instance we will concern ourselves with this case). Given such a target geometry, we fix the following discrete data: a degree $d \geq 0$, a number of marked points n and a matrix α of tangency orders. Here α is an $(r+1) \times n$ matrix of non-negative integers: the entry α_k^i for $i \in \{0, \ldots, r\}$ and $k \in \{1, \ldots, n\}$ records the tangency order of the marking x_k to the hyperplane H_i.

We will define log Gromov–Witten invariants with non-maximal tangency. This means that for each $i \in \{0, \ldots, r\}$ we only require that:

$$\sum_{k=1}^{n} \alpha_k^i \leq d$$

This is in contrast to [GS13, Che14, AC14], which require the above to be an equality.

3.2.2. **Definition of the moduli space**. Given the data above, we will define moduli spaces of relative stable maps to $(\mathbb{P}^N, D)$ in the spirit of Gathmann. This means that we will define the moduli space as a closed substack of the moduli space of absolute stable maps:

$$\overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha}(\mathbb{P}^N|D, d) \subseteq \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N, d)$$

The definition is as follows. For a fixed $i \in \{0, \ldots, r\}$, the ith row of the matrix α defines a vector of tangency conditions for the hypersurface H_i. We write $\alpha^i = (\alpha_k^i)_{k=1}^n$ for this vector. Then Gathmann defines in [Gat02] a moduli space of relative stable maps to H_i

$$\overline{\mathcal{M}}^{\mathrm{Gat}}_{0,\alpha^i}(\mathbb{P}^N|H_i, d) \subseteq \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N, d)$$

and proves that this space is irreducible, with dimension equal to the expected dimension:

$$\dim \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N, d) - \sum_{k=1}^{n} \alpha_k^i$$

Thus it has a natural fundamental class, which can be used to define relative Gromov–Witten invariants. The space is virtually birational to the other well-known moduli spaces of relative stable maps (see [Gat03a, Lemma 5.1.12] for a comparison to the space of maps to expanded degenerations), so we can compute the usual relative/log Gromov–Witten invariants of the smooth pair $(\mathbb{P}^N, H_i)$ by integrating against the fundamental class of the Gathmann space. (We note, however, that although the Gathmann space is irreducible, it is not in general smooth, and does not carry a natural perfect obstruction theory.)

Definition 3.2.1. Given $(\mathbb{P}^N, D)$ as above, the *snc moduli space of relative stable maps* is defined as the intersection of the Gathmann spaces with respect to each of the H_i:

$$\overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha}(\mathbb{P}^N|D, d) := \bigcap_{i=0}^{r} \overline{\mathcal{M}}^{\mathrm{Gat}}_{0,\alpha^i}(\mathbb{P}^N|H_i, d) \subseteq \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N, d)$$

This is a closed substack of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^N, d)$, with expected dimension:

$$\operatorname{vdim} \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N, d) - \sum_{i=0}^{r} \sum_{k=1}^{n} \alpha_k^i$$

We will see (Example 3.2.3 below) that, unlike with the case of a smooth divisor, this space is not necessarily irreducible or of the correct dimension. However, we can still define a *virtual* fundamental class as follows. By definition we have a Cartesian diagram:

$$\begin{array}{ccc} \overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha}(\mathbb{P}^N|D,d) & \hookrightarrow & \prod_{i=0}^r \overline{\mathcal{M}}^{\mathrm{Gat}}_{0,\alpha^i}(\mathbb{P}^N|H_i,d) \\ \downarrow & \square & \downarrow \\ \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d) & \xhookrightarrow{\Delta} & \prod_{i=0}^r \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d) \end{array}$$

Since $\mathbb{P}^N$ is convex $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d)$ is smooth, and so the diagonal Δ is a regular embedding. Hence there exists a Gysin map $\Delta^!$ [Ful98, §6]. We define

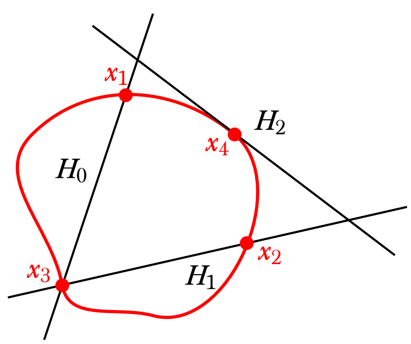
$$[\overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha}(\mathbb{P}^N|D,d)]^{\mathrm{virt}} := \Delta^!\left(\prod_{i=0}^r [\overline{\mathcal{M}}^{\mathrm{Gat}}_{0,\alpha^i}(\mathbb{P}^N|H_i,d)]\right)$$

and a quick check shows that this indeed has the correct dimension.

Example 3.2.2. Consider degree 2 maps to $(\mathbb{P}^2,(H_0+H_1+H_2))$ with 4 marked points and the following 3×4 matrix of tangency orders:

$$\alpha = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

A generic element of $\overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha}(\mathbb{P}^2|(H_0+H_1+H_2),2)$ can be drawn schematically as follows:



Note that this is also a perfectly valid (though no longer generic) element of any moduli space with strictly smaller tangencies than α (remember that, in the case of non-maximal tangency, the α^i_k only serve as *lower bounds* for the actual tangency orders). For instance, it belongs to the moduli space with the following tangency matrix:

$$\alpha' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \alpha - e_3^0 - 2e_4^2$$

This illustrates a general phenomenon: if $\alpha' \leq \alpha$ entrywise, then there is an inclusion

$$\overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha}(\mathbb{P}^N|D,d) \subseteq \overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha'}(\mathbb{P}^N|D,d)$$

which has virtual codimension:

$$\sum_{i=0}^{r}\sum_{k=1}^{n}\left(\alpha_k^i - (\alpha')_k^i\right)$$

Of course, the picture above is only of a generic element. As we move towards the boundary it can happen that entire components of the source curve are mapped into the divisor. In this case the condition to belong to the snc space is somewhat more complicated, though still entirely numerical (a consequence of the fact that we are working in genus zero). This was discussed in §3.1.1.

Example 3.2.3. When $D = H$ consists of a single hyperplane, the snc space is irreducible of the expected dimension [Gat02, Proposition 1.14], but this is no longer true if there is more than one hyperplane. To see why, let $D = \Sigma_{i=0}^r H_i$ and consider the *nice locus*

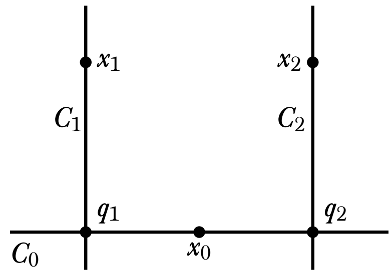
$$\mathcal{M}^{\circ}_{0,\alpha}(\mathbb{P}^N|D,d) \subseteq \overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha}(\mathbb{P}^N|D,d)$$

consisting of relative stable maps whose source curve is irreducible and not mapped inside D. This is a locally closed substack of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d)$, and it is not hard to show (using a mild generalisation of [Gat02, Lemma 1.8]) that it is irreducible of the correct dimension. If we take its closure inside the moduli space of ordinary stable maps, then the fact that the snc space is proper means that we have an inclusion of a closed subspace:

$$\overline{\mathcal{M}^{\circ}_{0,\alpha}(\mathbb{P}^N|D,d)} \subseteq \overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha}(\mathbb{P}^N|D,d)$$

This is in fact an irreducible component of the snc space, which we call the *main component.* We then observe that if the main component is not equal to all of $\overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha}(\mathbb{P}^N|D,d)$, then the latter is reducible.

Thus it remains to give an example in which the main component is not equal to the whole snc space; this amounts to finding a relative stable map which does not admit an infinitesimal deformation to an element of the nice locus. Consider $\mathbb{P}^2$ relative two hyperplanes H_0 and H_1, and consider a degree d stable map whose source curve is of the form



with C_0 mapped into $H_0 \cap H_1$ and C_1 and C_2 external components. We work in the situation of maximal tangency, and with $\alpha_0^0 = \alpha_0^1 = \alpha_0$ (these are the tangency conditions of x_0 with respect to H_0 and H_1). For $i \in \{0,1\}$ and $j \in \{1,2\}$ let m_j^i be the intersection multiplicity of C_j with H_i at $q_j = C_j \cap C_0$. Then the condition to belong to the snc space is:

$$m_1^0 + m_2^0 = \alpha_0^0 = \alpha_0$$
$$m_1^1 + m_2^1 = \alpha_0^1 = \alpha_0$$

(There is also a condition on the marked points x_1 and x_2, but this is not so important for our discussion.) Now, suppose for a contradiction that this stable map belongs to the closure of the nice locus. This means that there exists a smoothing

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathbb{P}^2 \\ \downarrow & & \\ \operatorname{Spec} R & & \end{array}$$

over a valuation ring R, with central fibre isomorphic to the stable map described above, and general fibre isomorphic to a stable map in the nice locus. For $i \in \{0,1,2\}$ let $\tilde{x}_i$ denote the section x_i on $\mathcal{C}$, viewed as a Cartier divisor. If we consider the Cartier divisors H_0 and H_1 on $\mathbb{P}^2$, then we find

$$f^*H_0 = \lambda_0 C_0 + \alpha_0 \tilde{x}_0 + \alpha_1^0 \tilde{x}_1 + \alpha_2^0 \tilde{x}_2$$
$$f^*H_1 = \lambda_1 C_0 + \alpha_0 \tilde{x}_0 + \alpha_1^1 \tilde{x}_1 + \alpha_2^1 \tilde{x}_2$$

for some $\lambda_0, \lambda_1 > 0$. Since the $\tilde{x}_i$ are Cartier, we see that C_0 is $\mathbb{Q}$-Cartier; then, since the central fibre is linearly trivial, it follows that $C_1 + C_2$ is $\mathbb{Q}$-Cartier. But since C_1 and C_2 are disjoint, this implies that C_1 and C_2 are each $\mathbb{Q}$-Cartier. In particular, we can make sense of the intersection numbers $C_i \cdot C_j \in \mathbb{Q}$ for any two components of the central fibre. Let $r_j = C_0 \cdot C_j \in \mathbb{Q}$ for

$j \in \{1,2\}$. Then for $i \in \{0,1\}$ and $j \in \{1,2\}$ we have:

$$\alpha^i_j x_j + m^i_j q_j = f^*(H_i) \cdot C_j = \alpha^i_j x_j + (\lambda_i r_j) q_j$$

Thus we obtain four equations

$$\lambda_0 r_1 = m^0_1 \qquad \lambda_1 r_1 = m^1_1$$
$$\lambda_0 r_2 = m^0_2 \qquad \lambda_1 r_2 = m^1_2$$

all involving positive rational numbers. Cross-multiplying, we obtain:

$$m^0_1 m^1_2 = m^1_1 m^0_2$$

But this is not always satisfied; remember that the only restriction we had was that

$$m^0_1 + m^0_2 = m^1_1 + m^1_2 = \alpha_0$$

so, for example, we can take:

$$m^0_1 = 1,\ m^0_2 = 3,\ m^1_1 = 2,\ m^1_2 = 2$$

This results in a degree 5 relative stable map, where C_1 has degree 2 and C_2 has degree 3, and with tangency matrix

$$\alpha = \begin{pmatrix} 4 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$$

which does not admit a smoothing. Thus, the corresponding moduli space is not irreducible.

Example 3.2.4. We can also show that the above moduli space is reducible via a dimension count. The expected dimension is

$$\text{vdim}\,\overline{\mathcal{M}}_{0,3}(\mathbb{P}^2, 5) - \Sigma^1_{i=0}\Sigma^3_{k=1}\alpha^i_k = (2 - 3 + 5 \cdot 3 + 3) - (4 + 1 + 4 + 1) = 7$$

which is also equal to the actual dimension of the main component (the closure of the nice locus). On the other hand, the locus of maps we have identified above is a product of moduli spaces corresponding to C_0, C_1 and C_2:

$$\overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,\alpha_{(1)}}(\mathbb{P}^2|(H_0 + H_1), 2) \times \overline{\mathcal{M}}_{0,\alpha_{(2)}}(\mathbb{P}^2|(H_0 + H_1), 3)$$

where the tangency matrices $\alpha_{(1)}$ and $\alpha_{(2)}$ are given by:

$$\alpha_{(1)} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \qquad \alpha_{(2)} = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$$

The expected dimension of this locus is then

$$\begin{aligned}\mathrm{vdim}\,\overline{\mathcal{M}}_{0,3} + \mathrm{vdim}\,\overline{\mathcal{M}}_{0,\alpha_{(1)}}(\mathbb{P}^2|(H_0+H_1),2) + \mathrm{vdim}\,\overline{\mathcal{M}}_{0,\alpha_{(2)}}(\mathbb{P}^2|(H_0+H_1),3) \\ = 0 + [(2-3+2\cdot 3+2)-(1+1+2)] + [(2-3+3\cdot 3+2)-(3+1+2)] \\ = 7\end{aligned}$$

and so the actual dimension of this locus is at least 7, which means in particular that it belongs to a component other than the main component.

3.3. Comparison of the moduli spaces

Having defined the snc moduli space in the previous section, it is natural to try and compare it to the moduli space of log stable maps. In this section we construct a natural morphism between these spaces, which we then use to probe their geometry. We will also see that the enumerative invariants defined using these spaces do not, in general agree. The main results can be summarised as follows:

Theorem 3.3.1. Consider a target geometry $(\mathbb{P}^N, D)$ as above, with discrete data d, n and α. Suppose that $\Sigma_{k=1}^n \alpha_k^i = d$ for all $i \in \{0, \ldots, r\}$, so that the corresponding moduli space of log stable maps is well-defined [GS13]. Then the map which forgets the log structures

$$\overline{\mathcal{M}}^{\mathrm{log}}_{0,\alpha}(\mathbb{P}^N|D,d) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d)$$

factors through the snc space, i.e. we have a map:

$$\tau : \overline{\mathcal{M}}^{\mathrm{log}}_{0,\alpha}(\mathbb{P}^N|D,d) \to \overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha}(\mathbb{P}^N|D,d)$$

This can be interpreted as a logarithmic desingularisation of the main component of the snc space, and coincides (up to a morphism which is bijective on geometric points) with an integralisation and saturation map. The map τ does not in general perserve the virtual classes (see Example 3.3.5).

3.3.1. **Factorisation through the snc space**. The first part of Theorem 3.3.1 is an easy consequence of the log stable map machinery.

Lemma 3.3.2. The map forgetting the log structures factors through the snc space.

Proof. This follows from the modified balancing condition for log maps [GS13, §1.4]. To be more precise: if $i \in \{0, \ldots, r\}$ and Z is an internal component with respect to H_i (i.e. Z is a maximal subcurve of $f^{-1}(H_i) \subseteq C$), then each irreducible component C' of Z produces an equation involving the tangency orders with respect to H_i (by projecting the modified balancing condition onto the factor of the monoid $\Gamma(C', f^*\overline{\mathcal{M}}_{(\mathbb{P}^N,D)})$ corresponding to H_i); if we sum these all together the contributions from the internal nodes cancel out, and we end up with precisely the condition that the stable map belongs to the Gathmann space wth respect to H_i. □

3.3.2. **Describing the forgetful morphism.** Thus we do indeed have a map

$$\tau : \overline{\mathcal{M}}^{\log}_{0,\alpha}(X|D,\beta) \to \overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha}(X|D,\beta)$$

and we would like to study its geometry. Note that these spaces have the same virtual dimension. Now, for each $i \in \{0,\ldots,r\}$ there is a natural inclusion of monoid sheaves on $\mathbb{P}^N$:

$$\mathcal{M}_{(\mathbb{P}^N,H_i)} \hookrightarrow \mathcal{M}_{(\mathbb{P}^N,D)}$$

Hence, there is a natural morphism of log schemes $(\mathbb{P}^N, D) \to (\mathbb{P}^N, H_i)$ which is the identity on the underlying scheme. By functoriality we obtain morphisms of moduli spaces

$$\overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^N|D,d) \to \overline{\mathcal{M}}^{\log}_{0,\alpha^i}(\mathbb{P}^N|H_i,d)$$

which fit into a commuting diagram:

$$\begin{array}{ccc}
\overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^N|D,d) & \longrightarrow & \prod_{i=0}^r \overline{\mathcal{M}}^{\log}_{0,\alpha^i}(\mathbb{P}^N|H_i,d) \\
\downarrow \tau & & \downarrow \prod_{i=0}^r \tau_i \\
\overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha}(\mathbb{P}^N|D,d) & \longrightarrow & \prod_{i=0}^r \overline{\mathcal{M}}^{\mathrm{Gat}}_{0,\alpha^i}(\mathbb{P}^N|H_i,d) \\
\downarrow & \square & \downarrow \\
\overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d) & \xrightarrow{\Delta} & \prod_{i=0}^r \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d)
\end{array}$$

We would very much like to say that the outer rectangle in this diagram is Cartesian. In fact this is *not true*, but for a rather subtle reason. The subtlety has to do with the formation of fibre products in different categories of log schemes. For the following discussion, we assume that the reader is familiar with Appendix 3.5.3, where these issues are discussed at length.

It is easy to show that the following diagram of log schemes is Cartesian

$$\begin{array}{ccc}
(\mathbb{P}^N, D) & \longrightarrow & \prod_{i=0}^r (\mathbb{P}^N, H_i) \\
\downarrow & \square & \downarrow \\
\mathbb{P}^N & \longrightarrow & \prod_{i=0}^r \mathbb{P}^N
\end{array}$$

where the log schemes in the bottom row are given the trivial log structure, i.e. $\mathcal{M}_{\mathbb{P}^N} = \mathcal{O}^*_{\mathbb{P}^N}$. This diagram is Cartesian in any of the following categories:

$$\mathbf{LogSch^{fs}} \hookrightarrow \mathbf{LogSch^{f}} \hookrightarrow \mathbf{LogSch^{coh}}$$

In particular, the fact that it is Cartesian in $\mathbf{LogSch^{fs}}$ means that we can apply [AC14, Theorem 2.6] and conclude that the corresponding diagram of moduli stacks

$$\begin{array}{ccc} \overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^N|D,d) & \longrightarrow & \prod_{i=0}^r \overline{\mathcal{M}}^{\log}_{0,\alpha^i}(\mathbb{P}^N|H_i,d) \\ \downarrow{\scriptstyle \tau} & \square & \downarrow{\scriptstyle \prod_{i=0}^r \tau_i} \\ \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d) & \xrightarrow{\Delta} & \prod_{i=0}^r \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d) \end{array}$$

is Cartesian *in the category of fs log stacks.* The crucial point, however, is that the forgetful functor

$$\mathbf{LogStacks^{fs}} \to \mathbf{Stacks}$$

does not preserve fibre products. To say it another way: the underlying stack of a fibre product of fs log stacks does not (in general) agree with the fibre product of the underlying stacks. Rather, the former is obtained from the latter by passing to a closed substack and then taking a finite cover (see Appendix 3.5.3).

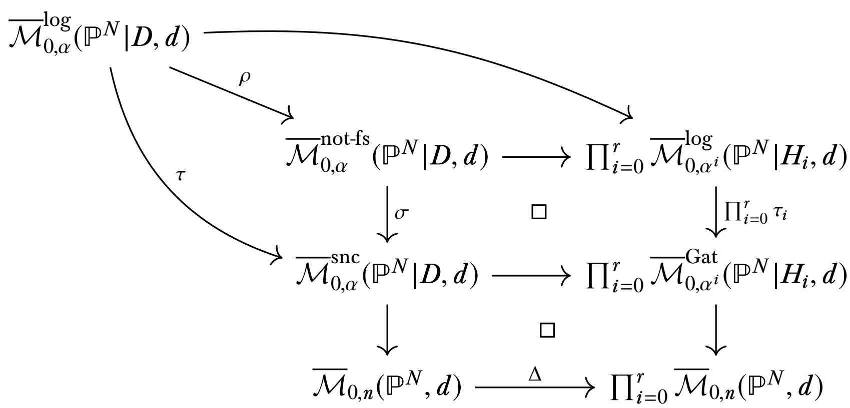
In order to understand this, let us denote by $\overline{\mathcal{M}}^{\text{not-fs}}_{0,\alpha}(\mathbb{P}^N|D,d)$ the fibre product in the category **Stacks** of ordinary stacks:

$$\begin{array}{ccc} \overline{\mathcal{M}}^{\text{not-fs}}_{0,\alpha}(\mathbb{P}^N|D,d) & \longrightarrow & \prod_{i=0}^r \overline{\mathcal{M}}^{\log}_{0,\alpha^i}(\mathbb{P}^N|H_i,d) \\ \downarrow{\scriptstyle \tau} & \square & \downarrow{\scriptstyle \prod_{i=0}^r \tau_i} \\ \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d) & \xrightarrow{\Delta} & \prod_{i=0}^r \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d) \end{array}$$

The notation is supposed to indicate that $\overline{\mathcal{M}}^{\text{not-fs}}_{0,\alpha}(\mathbb{P}^N|D,d)$ should be thought of as a moduli space of log stable maps where the base is no longer required to be fine or saturated. This has a natural log structure, which is not fs, and the integralisation and saturation process produces a map of log stacks

$$\rho\colon \overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^N|D,d) \to \overline{\mathcal{M}}^{\text{not-fs}}_{0,\alpha}(\mathbb{P}^N|D,d)$$

which on the underlying stacks is a finite cover of a closed substack. If we now forget about the log structures on the moduli spaces, we obtain a diagram of stacks:



This diagram will allow us to understand the morphism τ, and as such to compare the geometry of the snc space with the log space. First of all, it is known that the map $\prod_{i=0}^r \tau_i$ is finite [Che14, Proposition 3.7.5] (in fact, it is a bijection on geometric points [Ran17, Proof of Theorem A, Step III]). Therefore, the pulled-back map

$$\sigma \colon \overline{\mathcal{M}}^{\text{not-fs}}_{0,\alpha}(\mathbb{P}^N|D,d) \to \overline{\mathcal{M}}^{\text{snc}}_{0,\alpha}(\mathbb{P}^N|D,d)$$

is also finite. Thus, the geometry of these two spaces are not so different. In particular, their irreducible components are in bijection and so (see Example 3.2.3) both spaces will typically have components of excess dimension.

On the other hand, we claim that the space $\overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^N|D,d)$ in the top left is log smooth and irreducible of the correct dimension. Indeed, it has an obstruction theory [GS13, §5] over the moduli stack $\tilde{\mathfrak{M}}^{\log}_{0,n}$ of not-necessarily basic log smooth curves, given by

$$(\mathbf{R}^\bullet \pi_* f^* \operatorname{T}^{\log}_{\mathbb{P}^N|D})^\vee$$

and in this case the logarithmic tangent bundle fits into a logarithimic Euler sequence [Ang14, Theorem 4.3] [Dol07]:

$$0 \to \mathcal{O}_{\mathbb{P}^N} \to \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^N} \bigoplus_{i=r+1}^{N} \mathcal{O}_{\mathbb{P}^N}(1) \to \operatorname{T}^{\log}_{(\mathbb{P}^N,D)} \to 0$$

Pulling back along f and passing to the long exact cohomology sequence, we obtain

$$\operatorname{H}^1(C, f^* \operatorname{T}^{\log}_{(\mathbb{P}^N,D)}) = 0$$

and so the moduli space is unobstructed over $\tilde{\mathfrak{M}}^{\log}_{0,n}$. This latter space is not smooth, but it is log smooth since the natural map

$$\tilde{\mathfrak{M}}^{\log}_{0,n} = \operatorname{Log}(\mathfrak{M}_{0,n}) \to \mathfrak{M}_{0,n}$$

is log étale [ACGS17, Lemma 3.3.2] and $\mathfrak{M}_{0,n}$ is log smooth. Since it is log smooth, the locus where the log structure is trivial forms a dense open subset [Niz06, Proposition 2.6], and in this case it is easy to see that this coincides with the image of the open embedding:

$$\mathcal{M}_{0,n} \hookrightarrow \mathfrak{M}_{0,n} \hookrightarrow \tilde{\mathfrak{M}}^{\log}_{0,n}$$

Thus $\tilde{\mathfrak{M}}^{\log}_{0,n}$ is irreducible (since $\mathcal{M}_{0,n}$ is), and has dimension equal to $\dim \mathcal{M}_{0,n} = n-3$. Since the map

$$\overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^N|D,d) \to \tilde{\mathfrak{M}}^{\log}_{0,n}$$

is smooth, it follows that $\overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^N|D,d)$ is irreducible of the expected dimension. Moreover since the map is smooth and strict it is also log smooth, from which it follows that $\overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^N|D,d)$ is log smooth (note, however, that it is not in general smooth, since $\tilde{\mathfrak{M}}^{\log}_{0,n}$ has singularities).

Bringing these observations together, we see that we have a morphism

$$\tau\colon \overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^N|D,d) \to \overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha}(\mathbb{P}^N|D,d)$$

whose source is log smooth and whose target contains many components of excess dimension. Furthermore, τ is an isomorphism over the nice locus, where the log structure on C is simply obtained by pulling back the log structure from the target. Therefore the image of τ is equal to the main component of $\overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha}(\mathbb{P}^N|D,d)$, and τ itself can be interpreted as a log desingularisation of this main component.

Here we see the true power of the logarithmic approach. We knew from the beginning that the snc space had a main component, but we did not know how to identify it. The logarithmic machinery not only identifies this component, but also logarithmically desingularises it. In §3.4 we will see how to apply this in order to compute invariants.

3.3.3. **Connectedness of the moduli spaces**. Here we take a brief detour to record a simple but important fact (used implicitly in the discussion above).

Lemma 3.3.3. All of the following spaces are connected:

$$\overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^N|D,d) \qquad \overline{\mathcal{M}}^{\text{not-fs}}_{0,\alpha}(\mathbb{P}^N|D,d) \qquad \overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha}(\mathbb{P}^N|D,d)$$

Proof. We first show that the moduli space of log stable maps is connected. The argument we present is due to M. Gross (though we have since discovered that a similar argument has been given independently in [CS13, §2]). As already discussed, the moduli space

$$\overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^N|D,d)$$

can be considered as an fs log stack over the trivial log point, and our earlier arguments show that it is log smooth (alternatively, use the same argument as in the proof of [CS13, Proposition 2.1]). By [Niz06, Proposition 2.6] it follows that the locus on which the log structure is trivial is dense. Therefore, it suffices to show that this locus is connected.

The log structure is trivial if and only if the ghost sheaf is trivial, and the construction of the minimal monoid given in [GS13, §1.5] shows that this can only happen when the source curve is smooth and does not map into any of the components of D. But in this case the log stable map is uniquely determined by the underlying stable map, and so this locus is isomorphic to the nice locus in the snc space (see Example 3.2.3). As already discussed, one can easily show that the nice locus is connected (in fact, irreducible) because it admits an explicit parametrisation. Thus, we conclude that the log moduli space is connected.

Now, recall that the map

$$\overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^N|D,d) \to \overline{\mathcal{M}}^{\text{not-fs}}_{0,\alpha}(\mathbb{P}^N|D,d)$$

is given by integralisation and saturation of log stacks. Since this construction is carried out locally on the target, the number of connected components of the source must be greater than or equal to the number of connected components of the target; hence we conclude that $\overline{\mathcal{M}}^{\text{not-fs}}_{0,\alpha}(\mathbb{P}^N|D,d)$ is connected. Finally, the map

$$\overline{\mathcal{M}}^{\text{not-fs}}_{0,\alpha}(\mathbb{P}^N|D,d) \to \overline{\mathcal{M}}^{\text{snc}}_{0,\alpha}(\mathbb{P}^N|D,d)$$

is surjective (since it is obtained by pulling back a surjective morphism) and bijective on geometric points, and so we conclude that $\overline{\mathcal{M}}^{\text{snc}}_{0,\alpha}(\mathbb{P}^N|D,d)$ is also connected. □

Remark 3.3.4. It would be nice to give a direct proof that $\overline{\mathcal{M}}^{\text{snc}}_{0,\alpha}(\mathbb{P}^N|D,d)$ is connected, by showing that a general object of this space can always be deformed to lie in the nice locus. We have some ideas on how to go about this, but are not yet able to give a complete argument which covers all cases.

3.3.4. **Comparing the virtual classes**. It is natural to ask whether the map

$$\tau : \overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^N|D,d) \to \overline{\mathcal{M}}^{\text{snc}}_{0,\alpha}(\mathbb{P}^N|D,d)$$

preserves the virtual classes. In general, the answer is "no". Here we will give a simple example to demonstrate this fact. Note that since the source of τ is irreducible of the correct dimension, its virtual class is just the ordinary fundamental class.

Example 3.3.5. Consider the target geometry $(\mathbb{P}^2, H_0 + H_1)$, with $d = 2$, $n = 1$ and the following vector of tangency orders:

$$\alpha = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

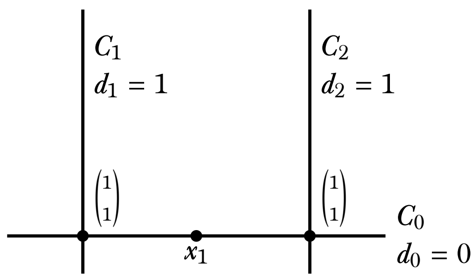
Consider the corresponding map of moduli spaces:

$$\tau\colon \overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^2|D,2) \to \overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha}(\mathbb{P}^2|D,2)$$

In this case the virtual dimension is 2. As already discussed, the logarithmic moduli space is irreducible of the correct dimension, and τ maps surjectively onto the main component of the snc space. We will now examine the geometry of the snc space directly. As with any snc space, it has a main component of the expected dimension (in this case, 2):

$$\overline{\mathcal{M}}^{\mathrm{main}}_{0,\alpha}(\mathbb{P}^2|D,2) \subseteq \overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha}(\mathbb{P}^2|D,2)$$

Now let us consider the locus of maps of the following form:



This locus is isomorphic (up to a double cover) to the product of the moduli spaces for C_0,C_1 and C_2 separately, which is:

$$\overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha'}(\mathbb{P}^2|D,1) \times \overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha'}(\mathbb{P}^2|D,1), \qquad \alpha' = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The locus therefore has dimension $0 + 1 + 1 = 2$, and as such forms an irreducible component of the moduli space, distinct from the main component; we will call this the *other component*:

$$\overline{\mathcal{M}}^{\mathrm{other}}_{0,\alpha}(\mathbb{P}^2|D,2) \subseteq \overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha}(\mathbb{P}^2|D,2)$$

We claim that in this example, these two components make up the entire moduli space. To see this, consider an arbitrary element $(C, x_1, f) \in \overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha}(\mathbb{P}^2|D,2)$ and let $C' \subseteq C$ denote the irreducible component of C contaning x_1. We will consider several cases. First, if C' is not mapped inside H_0 or H_1 then we must have $C = C'$ and then (C, x_1, f) belongs to the nice locus, hence to the main component.

Next, if C' is mapped inside $H_0 \cap H_1$ then $\deg f|_{C'} = 0$ and by stability there must be two non-contracted components of the curve adjacent to C'. Each of these could be mapped inside H_0 or H_1, but infinitesimally we can deform them outside of these hyperplanes. Thus, (C, x_1, f) belongs to the other component.

Finally, suppose C' is mapped inside H_0 but not inside H_1. Then we must have $\deg f|_{C'} = 2$ since x_1 must have tangency order 2 with respect to H_1. So $C = C'$ and we may deform the map to obtain an element of the nice locus. Hence, (C, x_1, f) belongs to the main component.

We thus conclude that the snc moduli space is made up of two irreducible components, each of dimension 2:

$$\overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha}(\mathbb{P}^2|D,2) = \overline{\mathcal{M}}^{\mathrm{main}}_{0,\alpha}(\mathbb{P}^2|D,2) \cup \overline{\mathcal{M}}^{\mathrm{other}}_{0,\alpha}(\mathbb{P}^2|D,2)$$

Thus we may write the virtual fundamental class as

$$[\overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha}(\mathbb{P}^2|D,2)]^{\mathrm{virt}} = \lambda \cdot [\overline{\mathcal{M}}^{\mathrm{main}}_{0,\alpha}(\mathbb{P}^2|D,2)] + \mu \cdot [\overline{\mathcal{M}}^{\mathrm{other}}_{0,\alpha}(\mathbb{P}^2|D,2)]$$

for some $\lambda, \mu \in \mathbb{Q}$. Moreover, since the virtual fundamental class is defined by taking a refined intersection product inside the (smooth) moduli space of absolute stable maps to $\mathbb{P}^2$, we have by [Ful98, §8.2] that $\lambda, \mu > 0$.

On the other hand, τ maps onto the main component of the snc space and is an isomorphism over the (dense) nice locus, and so:

$$\tau_*[\overline{\mathcal{M}}^{\mathrm{snc}}_{0,\alpha}(\mathbb{P}^2|D,2)] = [\overline{\mathcal{M}}^{\mathrm{main}}_{0,\alpha}(\mathbb{P}^2|D,2)]$$

In paticular, τ does not preserve the virtual fundamental classes: there is a non-zero piece of the virtual class of the snc space supported away from the main component, which is not seen by τ.

3.4. Towards a recursion formula

The original motivation for introducing the snc moduli spaces was to prove a Gathmann-like recursion formula for log Gromov–Witten invariants. However, as we saw in the previous section, the snc moduli spaces are not suited for computing log invariants, since they are not virtually birational to the moduli spaces of log stable maps.

This suggests that, in order to obtain a recursion formula for log invariants, we should work directly on the log space. This approach provides a number of simplifications – the spaces involved are irreducible of the correct dimension, and can be partially described using tropical geometry – but also presents its own difficulties, the most pressing of which is the question of gluing for log maps, i.e. understanding the recursive structure of the boundary of the moduli space.

In this section we will attempt to apply the recursion formula to a number of examples, explaining the difficulties we encounter and our attempts to overcome them. Our approach will be rather informal, and it should be clear to the reader that this is still very much work in progress.

3.4.1. **The recursion scheme**. We assume that the reader is familiar with Gathmann's recursion formula for smooth divisors, as described in §2.2.2. The basic idea of the snc recursion is as follows. We start with a moduli space of log stable maps:

$$\overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^N|D,d)$$

Choose a marked point $x_k \in \{x_1,\ldots,x_n\}$ and a hyperplane $H_j \in \{H_0,\ldots,H_r\}$ with $\alpha^j_k > 0$. This is where we will perform the recursion. Consider the following (larger) moduli space, obtained by lowering the tangency order at x_k:

$$\overline{\mathcal{M}}^{\log}_{0,\alpha-e^j_k}(\mathbb{P}^N|D,d)$$

As in [Gat02, Construction 2.1], we can construct a line bundle on this moduli space whose first Chern class is equal to

$$(\alpha^j_k-1)\cdot\psi_k+\mathrm{ev}^*_k H$$

and which carries a regular section whose vanishing locus includes the divisor

$$\overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^N|D,d)\subseteq\overline{\mathcal{M}}^{\log}_{0,\alpha-e^j_k}(\mathbb{P}^N|D,d)$$

as well as a number of *comb loci*, given by the strata in $\overline{\mathcal{M}}^{\log}_{0,\alpha-e^j_k}(\mathbb{P}^N|D,d)$ where x_k belongs to a component of the source curve mapped inside H_i. Consequently we obtain a relation in the Chow group of

$$\overline{\mathcal{M}}^{\log}_{0,\alpha-e^j_k}(\mathbb{P}^N|H,d)$$

which one should think of as describing how to decrease the tangency of x_k to H_j. The problem – and this is not only a technical problem, but actually involves some geometry – is to identify the comb loci which appear here, and to express integrals over them in a recursive manner.

We will see how this plays out in a number of examples. In order to identify the comb loci, we will adopt the following strategy. Consider the map

$$\varphi\colon \overline{\mathcal{M}}^{\log}_{0,\alpha-e^j_k}(\mathbb{P}^N|D,d) \to \overline{\mathcal{M}}^{\log}_{0,\alpha^j-e_k}(\mathbb{P}^N|H_j,d)$$

which we may think of as a closed embedding. The line bundle and section which we constructed also make sense on $\overline{\mathcal{M}}^{\log}_{0,\alpha^j-e_k}(\mathbb{P}^N|H_j,d)$, and as such we may obtain our comb loci by first identifying the appropriate comb loci in $\overline{\mathcal{M}}^{\log}_{0,\alpha^j-e_k}(\mathbb{P}^N|H_j,d)$ (by essentially the same analysis as in [Gat02]) and then pulling back along φ.

Remark 3.4.1. The reader familiar with log Gromov–Witten theory may feel somewhat uneasy hearing us talk about "decreasing the tangency", since log stable maps must, by definition, have maximal tangency, i.e. we always have

$$\sum_{i=1}^{n} \alpha^j_i = d$$

for each hyperplane H_j. The way we make sense of non-maximal tangencies such as $\alpha - e^j_k$ is to introduce an additional marked point (which we refer to as *fictitious*) which has tangency 1 with respect to H_j (and zero with respect to all other hyperplanes). Intuitively, there is not much difference between this space and the true "non-maximal tangency space", and this can actually be made precise [Gat02, Lemma 1.15(i)].

Notation 3.4.2. Through this section, we will use the shorthand notation

$$\Big[\alpha\Big]^{\mathbb{P}^N}_d$$

to denote the following moduli space of log stable maps:

$$\overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^N|D,d)$$

Here, as before, α is a matrix of tangency orders, with the columns of α corresponding to the marked points and the rows corresponding to the components of the divisor D. If $x_i \in \{x_1,\ldots,x_n\}$ is a marked point, we will use

$$H[i] := \mathrm{ev}_i^* H$$

to denote the induced cohomology class on $[\alpha]^{\mathbb{P}^N}_d$.

3.4.2. **Example 1**. Consider the moduli space

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}^{\mathbb{P}^2}_1$$

parametrising degree 1 maps to $\mathbb{P}^2$ with 5 marked points and tangency orders to the toric boundary as specified in the above matrix. This has dimension equal to the virtual dimension

$$\operatorname{vdim} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}^{\mathbb{P}^2}_1 = \operatorname{vdim} \overline{\mathcal{M}}_{0,5}(\mathbb{P}^2, 1) - 3 = 4$$

and so we can attempt to compute the following log Gromov–Witten invariant:

$$H[4]^2 \psi_4 \psi_5 \cap \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}^{\mathbb{P}^2}_1$$

This has been computed in [MR16, Example 5.5] by an enumeration of tropical curves, giving an answer of 2. Here we will compute this invariant using the log recursion.

The procedure, as outlined in §3.4.1 above, is to reduce the tangencies at each of the marked points. Let us start with x_1, which is tangent to H_1 to order 1 (and not tangent to any of the other hyperplanes). We must consider the corresponding recursion formula in the moduli space of log stable maps to $(\mathbb{P}^2, H_1)$

$$\overline{\mathcal{M}}^{\log}_{0,(0,\ldots,0)}(\mathbb{P}^2 | H_1, 1) = \overline{\mathcal{M}}_{0,5}(\mathbb{P}^2, 1)$$

which in this case takes a particularly simple form:

$$[\overline{\mathcal{M}}^{\log}_{0,(1,0,\ldots,0)}(\mathbb{P}^2 | H_1, 1)] = H[1] \cap [\overline{\mathcal{M}}^{\log}_{0,(0,\ldots,0)}(\mathbb{P}^2 | H_1, 1)]$$

The reason no comb loci appear here is because the equation

$$d_0 + \sum_{i=1}^{r} m_i = \sum_{x_k \in C_0} \alpha_k^j$$

is never satisfied, since the right hand side is zero. Pulling back along φ as in §3.4.1, we obtain

$$H[4]^2\psi_4\psi_5 \cap \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}_1^{\mathbb{P}^2} = H[1]H[4]^2\psi_4\psi_5 \cap \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}_1^{\mathbb{P}^2}$$

Applying the same argument to x_2 and x_3, we can remove all the tangencies and obtain:

$$H[4]^2\psi_4\psi_5 \cap \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}_1^{\mathbb{P}^2} = H[1]H[2]H[3]H[4]^2\psi_4\psi_5 \cap [\overline{\mathcal{M}}_{0,5}(\mathbb{P}^2, 1)]$$

The right hand side can be computed by applying the divisor equation, the dilaton equation and finally the Mirror Theorem (I used Growi [Gat]). The answer is 2, as expected.

3.4.3. **Example 2.** The previous example was easy because no comb loci appeared. However, this is usually too much to hope for; we will now present a simple case in which comb loci do in fact appear. Consider the following moduli space:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}_2^{\mathbb{P}^2}$$

This has dimension given by:

$$\operatorname{vdim} \overline{\mathcal{M}}_{0,3}(\mathbb{P}^2, 2) - 4 = 4$$

Let us try to apply the recursion to compute the following invariant:

$$\psi_3^4 \cap \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}_2^{\mathbb{P}^2}$$

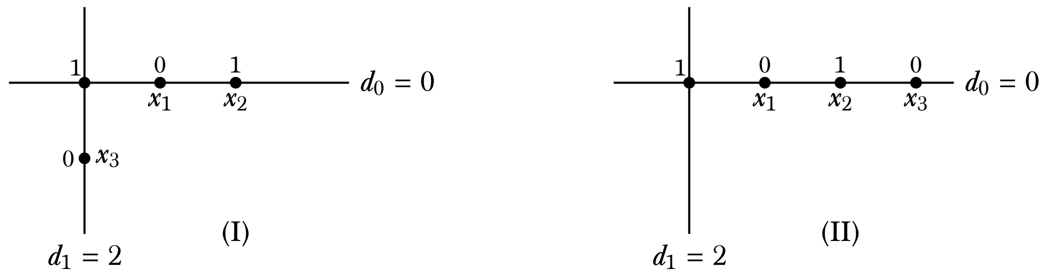
We begin by recursing at x_1 with respect to H_1. The first thing to do is to identify Gathmann's comb loci in the corresponding moduli space of log stable maps to $(\mathbb{P}^2, H_1)$:

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}_2^{\mathbb{P}^2}$$

(We will then have to pull these back to obtain the comb loci in the space of maps to $(\mathbb{P}^2, H_1 + H_2)$). Recall that we have an internal component C_0 containing x_1 and a collection $C_1, \ldots, C_r$ of external components such that:

$$d_0 + \sum_{i=1}^{r} m_i = \sum_{x_i \in C_0} \alpha_i \tag{3.4.1}$$

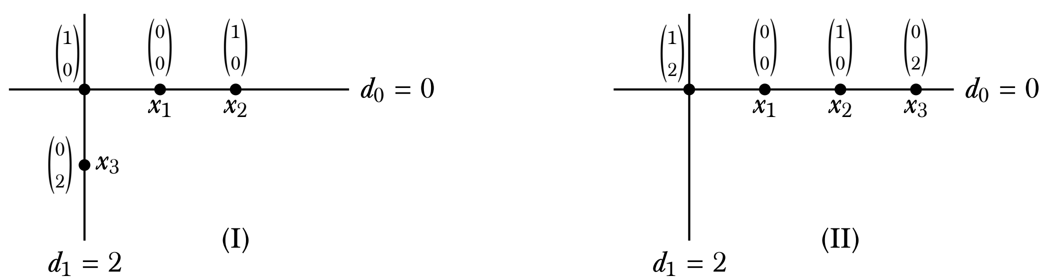
In this case, we must have that $x_2 \in C_0$ in order for the right hand side to be non-zero. With $x_2 \in C_0$, the right hand side is equal to 1, and so the only possibility is $d_0 = 0$ and $r = 1$ with $m_1 = 1$. There are two corresponding comb loci, given by:



Here we have indicated the tangencies with respect to H_1. Note that equation (3.4.1) is satisfied in both cases. These loci define divisors inside the moduli space

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^{\mathbb{P}^2}_2$$

which contribute to the recursion. Pulling back along φ corresponds to imposing the tangency conditions with respect to H_2. The loci we obtain are given by:



Note that C_0 is contracted inside H_1 in case (I) but inside $H_1 \cap H_2$ in case (II). The corresponding moduli spaces are equal to the products of the moduli spaces for C_0 and C_1:

$$\text{(I)}: \ \overline{\mathcal{M}}_{0,3} \times \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{\mathbb{P}^2}_2 \qquad\qquad \text{(II)}: \ \overline{\mathcal{M}}_{0,4} \times \begin{bmatrix} 1 \\ 2 \end{bmatrix}^{\mathbb{P}^2}_2$$

Note that these are both 4-dimensional, so they define divisors inside our original space

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{\mathbb{P}^2}_2$$

as expected. Now, locus (II) does not contribute because the ψ_3^4 insertion vanishes on $\overline{\mathcal{M}}_{0,4}$. On the other hand, locus (I) contributes

$$\psi_2^4 \cap \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}_2^{\mathbb{P}^2} = H[1]\psi_2^4 \cap \begin{bmatrix} 0 & 2 \end{bmatrix}_2^{\mathbb{P}^2} = -3/4$$

as calculated using Growi. Thus, the recursion formula reads:

$$\psi_3^4 \cap \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}_2^{\mathbb{P}^2} = H[1]\psi_3^4 \cap \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}_2^{\mathbb{P}^2} - (-3/4)$$

If we now recurse at x_2, no comb loci appear for the same reason as in Example 1 above. Thus we obtain:

$$\psi_3^4 \cap \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}_2^{\mathbb{P}^2} = H[1]H[2]\psi_3^4 \cap \begin{bmatrix} 0 & 0 & 2 \end{bmatrix}_2^{\mathbb{P}^2} + 3/4$$

The first term on the right hand side can now be computed using Gathmann's original recursion for relative Gromov–Witten invariants. The result (given by Growi) is:

$$\psi_3^4 \cap \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}_2^{\mathbb{P}^2} = 17/8 + 3/4 = 23/8$$

3.4.4. **Example 3**. In the previous example an important step was to identify the preimage of Gathmann's comb locus under the map:

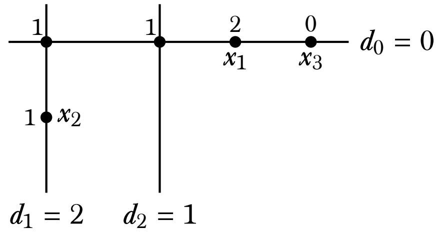
$$\overline{\mathcal{M}}_{0,\alpha}^{\log}((\mathbb{P}^2|H_1 + H_2), d) \xrightarrow{\varphi} \overline{\mathcal{M}}_{0,\alpha^1}^{\log}(\mathbb{P}^2|H_1, d)$$

In the example we considered, this was not terribly difficult: we just imposed the tangencies with respect to H_2 in the only way possible. However, in general the preimage can be more tricky to identify. We will illustrate this through two examples: the current one, which can be solved using tropical geometry, and a further example which requires more sophisticated methods. To keep the exposition simple, we will abandon the attempt at calculating individual invariants, and instead focus our attention on certain specific comb loci. Consider the following moduli space of log stable maps to $(\mathbb{P}^2, H_1 + H_2)$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}_3^{\mathbb{P}^2} \tag{3.4.2}$$

which has dimension 5. Let us recurse at x_1, but now by *increasing* the tangency. This means that the comb loci we consider consist of the subloci of the moduli space (3.4.2), in which x_1

belongs to an internal component of the curve. As usual, we begin by considering the comb loci with respect to H_1. In particular, we will focus our attention on the following locus



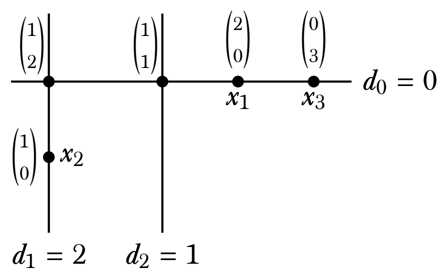
where as usual, the tangencies we have indicated are with respect to H_1. This gives a locus inside the moduli space

$$\begin{bmatrix} 2 & 1 & 0 \end{bmatrix}_3^{\mathbb{P}^2}$$

and in order to obtain the snc comb locus, we must identify its preimage along the map:

$$\varphi\colon \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}_3^{\mathbb{P}^2} \to \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}_3^{\mathbb{P}^2}$$

There is an obvious way to impose the tangencies with respect to H_2, illustrated in the following figure:



When we do this, however, we find that the corresponding moduli space is given by

$$\overline{\mathcal{M}}_{0,4} \times \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}_2^{\mathbb{P}^2} \times \begin{bmatrix} 1 \\ 1 \end{bmatrix}_1^{\mathbb{P}^2} \tag{3.4.3}$$

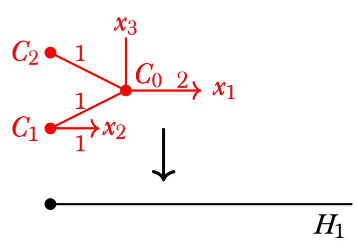
which has dimension $1 + 3 + 1 = 5$. But this was supposed to be a divisor inside our original moduli space

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}_3^{\mathbb{P}^2}$$

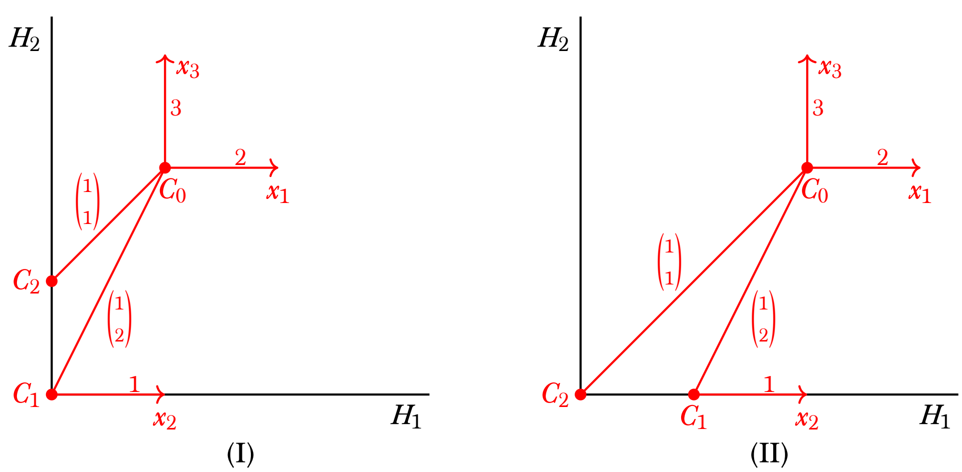
which also has dimension 5. So we have ended up with a locus of dimension 1 greater than expected. What has gone wrong?

The explanation is as follows. What we really have done, in imposing the tangencies with respect to H_2 as above, is to calculate the preimage of our comb locus in the *snc moduli space*. But, as we saw before, the snc moduli space can have many components of excess dimension. What we really want, therefore, is the preimage in the *log moduli space*, which will actually be much smaller; in fact, it will be equal (up to a finite map) to the intersection of the locus above with the main component of the snc space. How are we to identify this preimage? We do not yet have a fully satisfactory answer to this question, but we do have certain techniques at our disposal which in some cases (such as the present one) are enough to completely identify the preimage.

The first idea is to use the connection between log Gromov–Witten theory and tropical geometry. We know that any log stable map must induce a corresponding map on the tropicalisations. Consider the comb locus with respect to H_1 identified above. The corresponding tropical picture is as follows:



Now, consider the corresponding locus of log stable maps to $(\mathbb{P}^N, H_1)$. We are trying to identify the sublocus consisting of those maps which admit a lifting to a log stable map with target $(\mathbb{P}^N, H_1 + H_2)$. If such a lift exists, then it must tropicalise to give a tropical map lifting the one above. We see that there are two possibilities for such a tropical lift:



Thus, we are forced to put either C_1 or C_2 into a hyperplane in order to obtain a continuous tropical map. The corresponding moduli spaces are:

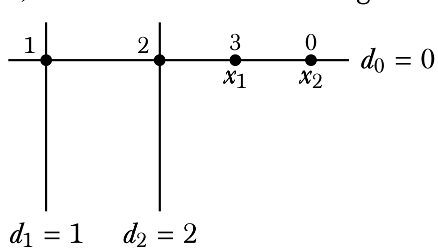
$$\text{(I):}\quad \overline{\mathcal{M}}_{0,4} \times \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}_2^{\mathbb{P}^2} \times \begin{bmatrix} 1 \end{bmatrix}_1^{\mathbb{P}^1} \qquad\qquad \text{(II):}\quad \overline{\mathcal{M}}_{0,4} \times \begin{bmatrix} 1 \\ 1 \end{bmatrix}_1^{\mathbb{P}^2} \times \begin{bmatrix} 2 & 0 \end{bmatrix}_2^{\mathbb{P}^1}$$

Each of these has dimension 4, and hence forms a divisor in the original moduli space (3.4.2); thus, we have succesfully identified the preimage of Gathmann's comb locus in the log moduli space. Notice that the 5-dimensional locus (3.4.3) which we identified earlier would correspond in the tropical picture to having both C_1 and C_2 mapped to the origin; but this does not give a valid tropical map (the resulting map is not continuous). Analysing the valid tropical maps gives us a precise description of the intersection of the locus (3.4.3) with the main component: it is the sublocus where either C_1 or C_2 is mapped into, respectively, H_1 or H_2. This is a description which we would have been unlikely to obtain without the logarithmic/tropical picture in mind.

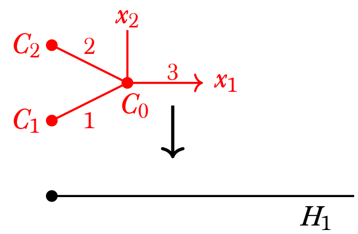
3.4.5. **Example 4.** The existence of a tropical lift is a necessary, but not in general sufficient, condition for the existence of a log lift. In this final example, we discuss a case in which the locus presented to us by the tropical picture has too high dimension, and must be cut down by additional conditions which guarantee the existence of a log lift. We consider the moduli space

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}_3^{\mathbb{P}^2}$$

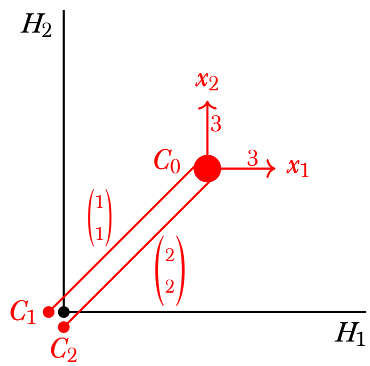
which one can check is 4-dimensional. Suppose that we are increasing the tangency order at x_1 with respect to H_1. In doing so, we encounter the following comb locus with respect to H_1:



As in the previous example, we will attempt to identify the corresponding locus in the space of maps to $(\mathbb{P}^2, H_1 + H_2)$ by way of tropical geometry. The tropical picture corresponding to the above locus is:



Now we want to impose the tangency conditions with respect to H_2. On the tropical level, this involves lifting the map above to a tropical curve mapping into the tropicalisation of $(\mathbb{P}^2, H_1+H_2)$. The most generic lift possible is the following



where both C_1 and C_2 are mapped to the origin. The corresponding moduli space of stable maps is equal to

$$\overline{\mathcal{M}}_{0,4} \times \begin{bmatrix} 1 \\ 1 \end{bmatrix}_1^{\mathbb{P}^2} \times \begin{bmatrix} 2 \\ 2 \end{bmatrix}_2^{\mathbb{P}^2} \tag{3.4.4}$$

which has dimension $1 + 1 + 2 = 4$. But the original moduli space was

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}_3^{\mathbb{P}^2}$$

which also has dimension 4. Thus, we have ended up with a locus of dimension 1 too large.

The issue is that in this case the existence of a tropical lift is not enough to guarantee the existence of a log lift. We are forced, therefore, to seek out additional criteria for log liftability. The basic idea which we will pursue here is that the existence of a log lift implies the existence of a rational function on C satisfying certain special properties. As we will see, the existence of this rational function will impose additional conditions on the moduli of C.

Suppose therefore that a log lift exists, and consider the following diagram of monoid sheaves on C:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_C^* & \longrightarrow & f^*\mathcal{M}_X & \longrightarrow & f^*\overline{\mathcal{M}}_X & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_C^* & \longrightarrow & \mathcal{M}_C & \longrightarrow & \overline{\mathcal{M}}_C & \longrightarrow & 0 \end{array}$$

The space of global sections of $f^*\overline{\mathcal{M}}_X$ is canonically identified with $\mathbb{N}^2$. Taking the section $(1,0)$ and restricting to the component $C_0 \subseteq C$, we obtain from the above diagram an induced isomorphism of line bundles on C_0:

$$f_0^*\mathcal{O}_{\mathbb{P}^2}(H_1) \xrightarrow{\cong} \mathcal{O}_{C_0}(3x_1 - q_1 - 2q_2)$$

where $f_0 = f|_{C_0}$. Similarly, taking $(0,1)$ we get an isomorphism:

$$f_0^*\mathcal{O}_{\mathbb{P}^2}(H_2) \xrightarrow{\cong} \mathcal{O}_{C_0}(3x_2 - q_1 - 2q_2)$$

But there is a natural isomorphism

$$\mathcal{O}_{C_0} \xrightarrow{\cong} f_0^*\mathcal{O}_{\mathbb{P}^2}(H_1 - H_2)$$

given by $f_0^*(s_1/s_2)$. Thus we get an induced isomorphism

$$\mathcal{O}_{C_0} \xrightarrow{\cong} \mathcal{O}_{C_0}(3x_1 - 3x_2)$$

i.e. a rational function r_0 on C_0 which vanishes to order 3 at x_1, has a pole of order 3 at x_2 and is invertible everywhere else.

We can carry out the same construction on C_1 and C_2, producing rational functions r_1 and r_2 which glue with r_0 to produce a global rational function r on C. Now, if we suppose that the maps $f|_{C_1}$ and $f|_{C_2}$ have been fixed, then r_1 and r_2 are also fixed. Therefore the values which r_0 must take at q_1 and q_2 are also fixed (note that r_0 is non-vanishing at these points, by the discussion above). The moduli point $(C_0, x_1, x_2, q_1, q_2) \in \overline{\mathcal{M}}_{0,4}$ must be such that the rational function r_0 exists, and (as we discuss below) this is a codimension 1 condition. Thus we obtain a 3-dimensional sublocus of the 4-dimensional locus (3.4.4), which is precisely the comb locus we were looking for.

If we want to identify the cohomology class that this locus defines (and we do, since we want to compute integrals over it), then by the principle of continuity we may assume that r_0 needs to take the same value at q_1 and q_2. We may view r_0 as a degree 3 map $\mathbb{P}^1 \to \mathbb{P}^1$, totally ramified at x_1 and x_2. If we fix q_1 there are then 3 choices for what q_2 can be, one of which is $q_2 = q_1$.

Thus we see that the locus of curves satisfying the rational function condition represents the class

$$3D_{q_1q_2} \in \mathrm{A}^1(\overline{\mathcal{M}}_{0,4})$$

which is certainly something we can integrate over. Thus, we have succesfully managed to identify the comb locus in this example.

The two steps taken here – using tropical curves to identify candidate loci, and then imposing rational function conditions on these loci – seem to provide a general solution to the problem of identifying the logarithmic comb loci. However, the final form which the recursion formula should take, as well as proofs of some of the claims made above, are still work in progress.

Appendix 3.5: Notes on log geometry

In this appendix we provide a short introduction to log geometry. Our approach is somewhat unorthodox, placing particular emphasis on those technical points which are important for our work, while ignoring many others. Since there already exist many good general references for log schemes, we feel that this decision is justified. The goal is to provide the reader with the necessary background for understanding the work presented in §§3.1–3.4.

A guide to the literature. The original reference for log geometry is [Kat89]. Other good introductions include [ACG+13] [Kat96, §2] [Cai00a] [Che14, Appendix A]. While all of these are well-written and cover most of the important examples, their relative brevity means that many crucial details and technicalities are omitted. For these, one must consult the comprehensive [Ogu] and the (unfinished) [Gil09], the latter of which has a wonderful knack for drawing the reader's attention to potentially dangerous technical aspects.

3.5.1. **Log schemes**.

3.5.1.1. *Monoids.* The basic algebraic structure in log geometry is the monoid. For us, a monoid is a set with a binary operation, satisfying all the axioms for an abelian group except perhaps the existence of inverses.

Example 3.5.1. Every abelian group is a monoid. The simplest monoid which is not a group is the monoid of natural numbers $\mathbb{N}$, with addition as the binary operation. If R is any (commutative) ring, then R can be considered as a monoid under multiplication.

Remark 3.5.2. Usually we will write the binary operation additively and denote the identity by 0. An important exception is when dealing with the multiplicative monoid of a ring (discussed above) in which case we will write the binary operation multiplicatively and denote the identity by 1.

There is a whole zoo of adjectives used to describe monoids satisfying various properties: integral, saturated, sharp, torsion free, toric, etc. The sheer length of this list can be a bit bewildering at first, but eventually all these notions become familiar and natural. To avoid burdening the reader unnecessarily, we will introduce such terms only as and when they are needed.

3.5.1.2. *Log structures.* We now come to the central notion of this section.

Definition 3.5.3. Let $(X, \mathcal{O}_X)$ be a locally ringed space. A *prelog structure* is a sheaf of monoids $\mathcal{M}_X$ on X together with a morphism of monoid sheaves

$$\alpha\colon \mathcal{M}_X \to \mathcal{O}_X$$

where $\mathcal{O}_X$ is viewed as a monoid sheaf under multiplication. A *log structure* is a prelog structure such that the induced map

$$\alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$$

is an isomorphism.

The rough intuition here is that $\mathcal{M}_X$ consists of "local logarithms" of certain functions on X, with α serving as the exponential map. In some cases a function has a unique local logarithm, in which case α is injective. In other cases, there may be many choices for the logarithm. An invertible function should always have a unique logarithm, which is precisely the condition that a prelog structure needs to satisfy in order to be a log structure. Note that in general α is neither injective nor surjective.

Remark 3.5.4. Of course, we are only ever interested in the case where $(X, \mathcal{O}_X)$ is a scheme. Nevertheless, the definition makes sense for any locally ringed space, and I believe it is clearer to state it in this level of generality, in order to emphasise that in some sense the log structure does not "see" the fact that the underlying space is a scheme. When dealing with log structures, it is important to be able to think a bit more topologically, which is what I have tried to convey in this definition. See §3.5.2.2 below for a more in-depth discussion of this (somewhat philosophical) point.

Definition 3.5.5. A *log scheme* consists of a scheme equipped with a log structure. In keeping with the established notation, we will denote the underlying scheme by $\underline{X}$, the sheaf of monoids by $\mathcal{M}_X$ and the entire package by $X = (\underline{X}, \mathcal{M}_X)$.

Remark 3.5.6. Strictly speaking, what we have defined above is a *Zariski* log scheme. In many situations it is necessary to take a finer topology, usually the étale topology. For the purposes of this exposition, however, we will ignore this extra layer of subtlety.

Example 3.5.7. Let $\Bbbk$ be any field and take $\underline{X} = \operatorname{Spec} \Bbbk$, so $\underline{X}$ is a point with structure sheaf $\Bbbk$. Let Q be any monoid and consider the morphism:

$$\begin{aligned} \alpha \colon Q \oplus \Bbbk^* &\to \Bbbk \\ (q, \lambda) &\mapsto \begin{cases} \lambda & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases} \end{aligned}$$

This is a morphism of monoids as long as Q is *sharp*, meaning that $q_1 + q_2 = 0$ only if $q_1 = q_2 = 0$ (this should be thought of as a convexity assumption). It is then clear that $\alpha^{-1}(\Bbbk^*)$ is isomorphic to $\Bbbk^*$ via α, so we have defined a log structure. We will denote this by $(\operatorname{Spec} \Bbbk, Q)$.

Example 3.5.8. Let $\underline{X}$ be a smooth variety and let $\underline{D} \subseteq \underline{X}$ be any hypersurface. Define a sheaf of monoids $\mathcal{M}_X$ on $\underline{X}$ as follows:

$$\Gamma(U, \mathcal{M}_X) = \left\{ f \in \Gamma(U, \mathcal{O}_X) : f|_{U \setminus D} \in \Gamma(U \setminus D, \mathcal{O}_X^*) \right\}$$

That is, $\mathcal{M}_X$ is the subsheaf of $\mathcal{O}_X$ consisting of those functions which are invertible outside of $\underline{D}$. We take α to be the inclusion of monoid sheaves:

$$\alpha \colon \mathcal{M}_X \hookrightarrow \mathcal{O}_X$$

This defines a log structure on $\underline{X}$, called the *divisorial log structure* associated to $\underline{D}$. The divisorial log structure plays a fundamental role in logarithmic Gromov–Witten theory, where it is used to keep track of the tangency orders of a stable map.

Example 3.5.9. For any monoid P, consider the monoid algebra $\Bbbk[P]$ and the associated scheme $\underline{X} = \operatorname{Spec} \Bbbk[P]$. The natural map $P \to \Bbbk[P]$ induces a morphism of monoid sheaves

$$\underline{P} \to \mathcal{O}_X$$

where $\underline{P}$ is the constant sheaf on $\underline{X}$. This is a prelog structure, but not a log structure in general. However we will shortly see how general machinery can be used to produce a minimal "associated log structure" on $\underline{X}$:

$$\underline{P}^{\mathrm{a}} \to \mathcal{O}_X$$

3.5.1.3. *Basic constructions for monoid sheaves.* Much of the discomfort caused by log geometry stems from a lack of fluency when dealing with monoids and their sheaves. Monoids do not form an abelian category, so the general techniques of homological algebra do not apply. Nevertheless, the categories of monoids and monoid sheaves are sufficiently nice in the sense that various desirable (co)limits exist. In the following proposition we catalogue the ones which are most useful for our purposes, and provide references for the curious reader. In all cases, the proof proceeds by first recognising that the desired (co)limit exists in the category of monoids, and then sheafifying the presheaf obtained by applying this (co)limit on each open set.

Proposition 3.5.10. Let $\underline{X}$ be a topological space. Then the following (co)limits exist in the category of monoid sheaves on $\underline{X}$:

(1) Coequalisers (hence in particular cokernels) [Ogu, p.5].
(2) Fibred products [Ogu, §1.1] [Cai00b].
(3) Fibred coproducts [Ogu, (1.1.1)].

There is another universal construction worth mentioning. Given any monoid P, there is a universal associated abelian group P^{gp} with a morphism $P \to P^{\mathrm{gp}}$ such that any map with source P and target an abelian group factors through this morphism. Sheafifying this, we obtain:

Proposition 3.5.11. Let $\underline{X}$ be a topological space. Then the inclusion morphism from the category of abelian sheaves to the category of monoid sheaves admits a left adjoint, called *groupification*: $\mathcal{M} \mapsto \mathcal{M}^{\mathrm{gp}}$.

3.5.1.4. *Basic constructions for log schemes.* Now that we have some (co)limits of monoid sheaves at our disposal, we can use them to carry out some basic constructions involving log structures.

Definition-Lemma 3.5.12 (Log structure associated to a prelog structure). Given a prelog structure $\mathcal{N}_X \to \mathcal{O}_X$ on $\underline{X}$, there is a unique *associated log structure*

$$\mathcal{N}_X^{\mathrm{a}} \to \mathcal{O}_X$$

together with a morphism of prelog structures $\mathcal{N}_X \to \mathcal{N}_X^{\mathrm{a}}$ such that any morphism from $\mathcal{N}_X$ to a log structure factors uniquely through this one.

Proof. We define $\mathcal{N}_X^{\mathrm{a}}$ as the fibred coproduct in the category of monoid sheaves:

$$\begin{array}{ccc} \alpha^{-1}(\mathcal{O}_X^*) & \longrightarrow & \mathcal{N}_X \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{O}_X^* & \longrightarrow & \mathcal{N}_X^{\mathrm{a}} \end{array}$$

The universal property produces a map $\mathcal{N}_X^{\mathrm{a}} \to \mathcal{O}_X$ and it is easy to check that this prelog structure is in fact a log structure □

Example 3.5.13. In Example 3.5.9 above, we constructed for any monoid P a tautological prelog structure

$$\underline{P} \to \mathcal{O}_X$$

on $\underline{X} = \operatorname{Spec} \Bbbk[P]$. Applying the above construction, we obtain a canonical associated log structure:

$$\underline{P}^{\mathrm{a}} \to \mathcal{O}_X$$

This log scheme is typically denoted $\operatorname{Spec}(P \to \Bbbk[P])$ and is fundamental to the theory of charts, discussed in §3.5.2. Note that $\underline{P}^{\mathrm{a}}$ is almost never a constant sheaf, even though $\underline{P}$ is.

Definition 3.5.14 (Ghost sheaf). Given a log scheme $X = (\underline{X}, \mathcal{M}_X)$ the *ghost sheaf* (sometimes also called the *characteristic sheaf*) is denoted and defined

$$\overline{\mathcal{M}}_X := \mathcal{M}_X / \alpha^{-1}(\mathcal{O}_X^*)$$

i.e. as the cokernel of the monoid sheaf morphism $\alpha^{-1}(\mathcal{O}_X^*) \hookrightarrow \mathcal{M}_X$.

The ghost sheaf is a constructible sheaf, i.e. there exists a stratification of $\underline{X}$ into locally closed subsets such that $\overline{\mathcal{M}}_X$ is constant on each stratum. As such, the ghost sheaf can be understood entirely in terms of its stalks (including stalks at generic points) and generisation maps. One should think of the ghost sheaf as capturing the combinatorial information carried by the log structure. It is a fundamental tool in log geometry, and is the basis for the close interplay with tropical geometry.

Example 3.5.15. Consider Example 3.5.13 above with $P = \mathbb{N}^2$. We have $\underline{X} = \mathbb{A}^2$ with log structure $\mathcal{M}_X$ induced by the prelog structure which is given on global sections by:

$$\mathbb{N}^2 \to \Bbbk[x, y]$$
$$(a, b) \mapsto x^a y^b$$

This is nothing but the divisorial log structure with respect to the divisor $\underline{D} = \{xy = 0\} \subseteq \underline{X}$ (though we will not use this fact). Let us now describe the stalks of $\overline{\mathcal{M}}_X$. Recall that by construction the monoid sheaf $\mathcal{M}_X$ is given by

$$\mathcal{M}_X = \underline{\mathbb{N}}^2 \oplus_{\alpha^{-1}(\mathcal{O}_X^*)} \mathcal{O}_X^*$$

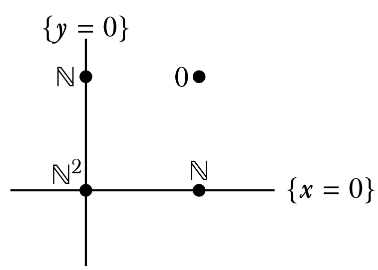
where α is the map $\underline{\mathbb{N}}^2 \to \mathcal{O}_X$. A basic (and quite believable) fact of category theory [Gil09, Lemma 4.3.4] then gives

$$\overline{\mathcal{M}}_X = \underline{\mathbb{N}}^2 / \alpha^{-1}(\mathcal{O}_X^*)$$

so in order to determine to stalks of $\overline{\mathcal{M}}_X$ we just need to determine the stalks of $\alpha^{-1}(\mathcal{O}_X^*)$. First take a point $p \in \underline{X}$ away from the two co-ordinate axes. The point $(a, b) \in \mathbb{N}^2$ is sent to $x^a y^b \in \mathcal{O}_{X,p}$ which is invertible since p does not lie on the co-ordinate axes. Thus, $\alpha^{-1}(\mathcal{O}_{X,p}^*) = \mathbb{N}^2$ and so the stalk $\overline{\mathcal{M}}_{X,p}$ is zero.

Next, take a point $p \in \underline{X}$ which belongs to the $\{x = 0\}$ axis but not to the $\{y = 0\}$ axis. Then $x^a y^b \in \mathcal{O}_{X,p}$ is invertible if and only if $a = 0$, so $\alpha^{-1}(\mathcal{O}_{X,p}^*) = \mathbb{N}$ and is embdedded into $\mathbb{N}^2$ as the second summand, so $\overline{\mathcal{M}}_{X,p} = \mathbb{N}$. The same argument applies for a point p belonging to the $\{y = 0\}$ axis but not to the $\{x = 0\}$ axis.

Finally, let $p \in \underline{X}$ be the origin. Then $x^a y^b \in \mathcal{O}_{X,p}$ is not invertible unless $a = b = 0$ and so $\overline{\mathcal{M}}_{X,p} = \mathbb{N}^2$. We thus obtain a complete picture of the stalks of $\overline{\mathcal{M}}_X$:



In fact, this gives a complete description of the sheaf, if we also include the generisation maps. Let η be the generic point of $\underline{X}$ and let σ_x and σ_y be the generic points of the co-ordinate axes. Let $p_0 = 0 \in \underline{X}$ be the origin. Then the constructible sheaf $\overline{\mathcal{M}}_X$ is described by the following diagram of stalks and generisation maps:

$$\begin{array}{ccccc} & & \overline{\mathcal{M}}_{X,\sigma_x} = \mathbb{N} & & \\ & \overset{\pi_1}{\nearrow} & & \searrow & \\ \overline{\mathcal{M}}_{X,p_0} = \mathbb{N}^2 & & & & \overline{\mathcal{M}}_{X,\eta} = 0 \\ & \underset{\pi_2}{\searrow} & & \nearrow & \\ & & \overline{\mathcal{M}}_{X,\sigma_y} = \mathbb{N} & & \end{array}$$

The monoid $\Gamma(X, \overline{\mathcal{M}}_X)$ of global sections is given by the limit of this diagram, which in this case is just $\mathbb{N}^2$, the stalk at the origin p_0. Note in particular that:

$$\mathbb{N}^2 = \Gamma(X, \overline{\mathcal{M}}_X) \neq \Gamma(X, \mathcal{M}_X)/\Gamma(X, \mathcal{O}_X^*) = \Bbbk^*/\Bbbk^* = 0$$

Definition-Lemma 3.5.16 (Fibred coproduct of log structures). Let $\underline{X}$ be a scheme and suppose we have a diagram of log structures on $\underline{X}$:

$$\begin{array}{ccc} \mathcal{N} & \longrightarrow & \mathcal{M}_1 \\ \downarrow & & \\ \mathcal{M}_2 & & \end{array}$$

Then there exists a fibred coproduct (pushout) $\mathcal{M}_1 \oplus_{\mathcal{N}} \mathcal{M}_2$ in the category of log structures on $\underline{X}$.

Proof. By Proposition 3.5.10 we may form the pushout $\mathcal{P}$ in the category of monoid sheaves, and by the universal property the dashed arrow in the following diagram exists:

$$\begin{array}{ccccc} \mathcal{N} & \longrightarrow & \mathcal{M}_1 & & \\ \downarrow & \lrcorner & \downarrow & \searrow & \\ \mathcal{M}_2 & \longrightarrow & \mathcal{P} & & \\ & \searrow & & \overset{-\,-\,\to}{\searrow} & \downarrow \\ & & & \longrightarrow & \mathcal{O}_X \end{array}$$

Using the fact that $\mathcal{N}, \mathcal{M}_1$ and $\mathcal{M}_2$ are log structures, one can show that $\mathcal{P} \to \mathcal{O}_X$ is a log structure. We leave it to the reader to check that this satisfies the required universal property. □

Definition 3.5.17 (Pull-back of log structures). Let $Y = (\underline{Y}, \mathcal{M}_Y)$ be a log scheme and let $f : \underline{X} \to \underline{Y}$ be a scheme morphism. We define a log structure $f^*\mathcal{M}_Y$ on $\underline{X}$, called the ***pull-back***

log structure, as follows. We have morphisms of monoid sheaves on $\underline{X}$

$$f^{-1}\mathcal{M}_Y \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$$

whose composition defines a prelog structure on $\underline{X}$. By definition $f^*\mathcal{M}_Y$ is then the log structure associated to this prelog structure. Its monoid sheaf is given by:

$$f^*\mathcal{M}_Y = f^{-1}\mathcal{M}_Y \oplus_{\alpha^{-1}(\mathcal{O}_X^*)} \mathcal{O}_X^*$$

In the next section we will define what it means to give a morphism of log schemes. We will then see that there is a natural map of log schemes $(\underline{X}, f^*\mathcal{M}_Y) \to (\underline{Y}, \mathcal{M}_Y)$, and that moreover this map is *strict* (see Definition 3.5.20).

3.5.1.5. *The category of log schemes.*

Definition 3.5.18. A morphism $f = (f, f^\flat)\colon (\underline{X}, \mathcal{M}_X) \to (\underline{Y}, \mathcal{M}_Y)$ of log schemes consists of a scheme morphism $f\colon \underline{X} \to \underline{Y}$ together with a morphism $f^\flat$ of monoid sheaves commuting with the structure maps:

$$\begin{array}{ccc} f^{-1}\mathcal{M}_Y & \overset{f^\flat}{\dashrightarrow} & \mathcal{M}_X \\ {\scriptstyle f^{-1}\alpha_Y}\downarrow & & \downarrow{\scriptstyle \alpha_X} \\ f^{-1}\mathcal{O}_Y & \xrightarrow{f^\sharp} & \mathcal{O}_X \end{array}$$

Thus a morphism of log schemes consists of strictly more data than a morphism of schemes. On the other hand, not every morphism of schemes admits a lift to a morphism of log schemes; we will see examples of this phenomenon in §3.4. As above, we will often abuse notation, using f to denote both a morphism of log schemes as well as the underlying morphism between the underlying schemes.

Remark 3.5.19. By the universal property of the associated log structure, giving a map $f^\flat$ as above is equivalent to giving a map of monoid sheaves making the following diagram commute

$$\begin{array}{ccccc} f^*\mathcal{M}_Y & & \overset{f^\flat}{\dashrightarrow} & & \mathcal{M}_X \\ & \searrow & & \swarrow & \\ & & \mathcal{O}_X & & \end{array}$$

i.e. a morphism of log structures $f^*\mathcal{M}_Y \to \mathcal{M}_X$ on $\underline{X}$.

Thus log schemes form a category, which we denote **LogSch**. There is an obvious forgetful morphism:

$$\mathbf{LogSch} \to \mathbf{Sch}$$

When considering moduli stacks of logarithmic objects, it is important to specify which of these categories the stack is fibred over. We prefer to consider stacks over **Sch**, since this is where we have the most tools at our disposal. On the other hand, there is a beautiful and non-trivial interplay between stacks over **LogSch** and log stacks over **Sch**, which unfortunately we do not have the time to enter into (the interested reader should consult [Kat00, §§3-4] [Gil12] [RSW17, §2.7]).

Definition 3.5.20. A log morphism $f\colon (\underline{X}, \mathcal{M}_X) \to (\underline{Y}, \mathcal{M}_Y)$ is called *strict* if the sheaf map $f^*\mathcal{M}_Y \to \mathcal{M}_X$ is an isomorphism.

Definition-Lemma 3.5.21 (Fibre product of log schemes). Suppose that we have a diagram of log schemes:

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z \end{array}$$

Then there exists a fibre product $X \times_Z Y$ in the category **LogSch**.

Proof. The underlying scheme of $X \times_Z Y$ is taken to be the schematic fibre product:

$$\begin{array}{ccc} \underline{X} \times_{\underline{Z}} \underline{Y} & \xrightarrow{\pi_X} & \underline{X} \\ {\scriptstyle \pi_Y}\downarrow & \square & \downarrow{\scriptstyle f} \\ \underline{Y} & \xrightarrow[g]{} & \underline{Z} \end{array}$$

To define the log structure, let π denote the projection $\underline{X} \times_{\underline{Z}} \underline{Y} \to \underline{Z}$. Pulling back the log structure maps

$$f^*\mathcal{M}_Z \to \mathcal{M}_X \qquad g^*\mathcal{M}_Z \to \mathcal{M}_Y$$

along π_X and π_Y respectively, we obtain a diagram of log structures on $\underline{X} \times_{\underline{Z}} \underline{Y}$:

$$\begin{array}{ccc} \pi^*\mathcal{M}_Z & \longrightarrow & \pi_X^*\mathcal{M}_X \\ \downarrow & & \\ \pi_Y^*\mathcal{M}_Y & & \end{array}$$

We define $\mathcal{M}_{X\times_Z Y}$ as the pushout of the above diagram in the category of log structures on $\underline{X} \times_{\underline{Z}} \underline{Y}$. We thus obtain a log scheme

$$X \times_Z Y = (\underline{X} \times_{\underline{Z}} \underline{Y}, \mathcal{M}_{X\times_Z Y})$$

which one can check satisfies the required universal property. □

The construction above shows that the map **LogSch** $\to$ **Sch** preserves fibre products. However, there is a snag: in the cases of interest to us, we will have the need to take fibre products not in the category **LogSch** of all log schemes, but in some full subcategory consisting of log schemes satisfying certain extra conditions (to be precise, we will be interested in the category of "fine and saturated" log schemes). The fibre products in these two categories do *not* agree in general, and moreover the fibre product in the smaller category does not commute with the fibre product in the category of schemes. This point, though technical, will be crucial for our study, and we will say a lot more about it in §3.5.3.

3.5.2. **Charts and coherence.** Up until this point, we have allowed $\mathcal{M}_X$ to be an essentially arbitrary sheaf of monoids. But we know, from our experience of working with sheaves of $\mathcal{O}_X$-modules, that arbitrary sheaves can be quite badly behaved, and that it is better to restrict ourselves to (quasi)coherent sheaves, which admit a nice local description. In this section we will discuss the corresponding notion in log geometry, that of a *(quasi)coherent log structure*, and highlight some of the similarities and important differences with (quasi)coherent sheaves.

3.5.2.1. *Charts.* Throughout this section we let $X = (\underline{X}, \mathcal{M}_X)$ denote a log scheme.

Definition 3.5.22. Let $\underline{U} \subseteq \underline{X}$ be an open set. Then a *local chart* for $(\underline{X}, \mathcal{M}_X)$ (on $\underline{U}$) consists of a monoid P and a monoid morphism

$$P \to \Gamma(U, \mathcal{M}_X)$$

such that the induced morphism of log structures on $\underline{U}$ is an isomorphism:

$$\underline{P}^{\mathrm{a}} \xrightarrow{\cong} \mathcal{M}_X|_U$$

Here the log structure $\underline{P}^{\mathrm{a}}$ is defined as follows. The monoid map above induces a morphism of monoid sheaves

$$\underline{P} \to \mathcal{M}_X|_U$$

where $\underline{P}$ is the constant sheaf on $\underline{U}$. Composing with the structure morphism $\mathcal{M}_X|_U \to \mathcal{O}_U$ makes $\underline{P}$ into a prelog structure, and the above map into a morphism of prelog structures (whose target just happens to be a log structure). Taking the associated log structure $\underline{P}^{\mathrm{a}}$ we obtain by the universal property a morphism of log structures $\underline{P}^{\mathrm{a}} \to \mathcal{M}_X|_U$.

There is an alternative way to describe the data of a chart, which is quite useful in practice.

Proposition 3.5.23. Let $(\underline{X}, \mathcal{M}_X)$ be a log scheme and $\underline{U} \subseteq \underline{X}$ an open set. Then the following data are equivalent:

(1) a local chart for $(\underline{X}, \mathcal{M}_X)$ on $\underline{U}$;

(2) a monoid P and a strict morphism of log schemes:

$$(\underline{U}, \mathcal{M}_X|_U) \to \operatorname{Spec}(P \to \Bbbk[P])$$

Moreover, these both induce a monoid map $P \to \Gamma(U, \mathcal{O}_U)$ whose associated log structure is isomorphic to $\mathcal{M}_X|_U$.

Proof. This is a simple exercise with definitions. □

Definition 3.5.24. A log scheme $(\underline{X}, \mathcal{M}_X)$ is called *quasicoherent* if it can be covered by open sets, each of which admits a chart. It is called *coherent* if on each open set the monoid P can be chosen to be finitely generated.

Almost all log schemes which appear in nature are coherent. Coherent log schemes are much easier to work with than arbitrary ones, since most local questions can be reduced to questions in algebra. However, some caution is required; there is an important respect in which coherent log schemes are much more badly-behaved than coherent sheaves; see §3.5.2.2 below.

Example 3.5.25 (Continuing Example 3.5.7). As before, let $\underline{X} = \operatorname{Spec} \Bbbk$ with log structure associated to a monoid Q. Since $\underline{X}$ is a point, a sheaf can be identified with its space of global sections and hence there is a tautological global chart:

$$Q \oplus \Bbbk^* \to \Gamma(X, \mathcal{M}_X) = Q \oplus \Bbbk^*$$

This chart, however, is not finitely generated. There is a better chart, given by:

$$\begin{aligned} Q &\to Q \oplus \Bbbk^* = \Gamma(X, \mathcal{M}_X) \\ q &\mapsto (q, 1) \end{aligned}$$

To verify that this is a chart, we consider the composition $Q \to Q \oplus \Bbbk^* \to \Bbbk$, which is given by:

$$\begin{aligned} \alpha \colon Q &\to \Bbbk \\ q &\mapsto \begin{cases} 1 & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases} \end{aligned}$$

The associated log structure is obtained via the fibre product

$$Q \oplus_{\alpha^{-1}(\Bbbk^*)} \Bbbk^* = Q \oplus \Bbbk^*$$

which is indeed isomorphic to $\mathcal{M}_X$. Thus, $(\operatorname{Spec} \Bbbk, Q)$ is always quasicoherent, and is coherent as long as Q is finitely generated.

Example 3.5.26 (Continuing Example 3.5.8). Consider a special case of the divisorial log structure, namely when $\underline{X}$ and $\underline{D}$ are both smooth. We will construct local charts for $(\underline{X}, \mathcal{M}_X)$. Choose an open cover of $\underline{X}$ which trivialises the line bundle $\mathcal{O}_{\underline{X}}(\underline{D})$. Given an open set $\underline{U}$ of this cover, let $f \in \Gamma(U, \mathcal{O}_X)$ be the equation cutting out $\underline{D}$. Consider the following monoid morphism:

$$\begin{aligned} \alpha : \mathbb{N} &\to \Gamma(U, \mathcal{M}_X) \subseteq \Gamma(U, \mathcal{O}_X) \\ 1 &\mapsto f \end{aligned}$$

Observe that $\alpha^{-1}\mathcal{O}_U^*$ is a constructible sheaf, equal to 0 along $\underline{D}$ and $\mathbb{N}$ elsewhere. Hence the stalks of $\underline{\mathbb{N}}^{\mathrm{a}}$ are equal to $\mathbb{N} \oplus \mathcal{O}_{U,x}^*$ along $\underline{D}$ and $\mathcal{O}_{U,x}^*$ elsewhere. Thus we see that the induced sheaf morphism

$$\underline{\mathbb{N}}^{\mathrm{a}} \to \mathcal{M}_X|_U$$

is an isomorphism on stalks, and hence is an isomorphism. Thus we have defined a chart, and so $(\underline{X}, \mathcal{M}_X)$ is coherent.

The assumptions that $\underline{X}$ and $\underline{D}$ be smooth are not necessary. The same argument as above works for any Cartier divisor $\underline{D}$ in any variety $\underline{X}$, and more generally we may consider normal crossings divisors $\underline{D}$ in $\underline{X}$, though in this case the monoid defining our chart will not be as simple as $\mathbb{N}$.

Example 3.5.27 (Continuing Example 3.5.13). If P is any monoid, then $\operatorname{Spec}(P \to \Bbbk[P])$ is quasicoherent by definition. If P is finitely generated then it is coherent, which happens if and only if the scheme $\operatorname{Spec} \Bbbk[P]$ is of finite type).

3.5.2.2. *Coherent log schemes do not deserve their name.* Having introduced coherence, I want to pause in order to discuss an important technical point, which in my opinion has not been given the attention it deserves in the literature.

In ordinary algebraic geometry, coherent sheaves admit a purely algebraic description, at least affine-locally: the category of coherent sheaves on an affine scheme $\underline{X} = \operatorname{Spec} A$ is equivalent to the category of finitely-generated A-modules.

Since this comes as second nature to most algebraic geometers, we often forget that it is not entirely tautological. A priori, if we are given a coherent sheaf $\mathcal{F}$ on an affine scheme $\underline{X} = \operatorname{Spec} A$, all we know is that there exists an affine open covering $\{\underline{X}_i = \operatorname{Spec} A_i\}_{i \in I}$ of $\underline{X}$, such that for each i the restriction $\mathcal{F}|_{X_i}$ corresponds to a finitely-generated A_i-module M_i. The non-trivial fact is that these modules M_i can in some way be "glued" to produce a global A-module M corresponding to $\mathcal{F}$.

There is an analogous situation in log geometry. Suppose we are given a coherent log scheme $(\underline{X}, \mathcal{M}_X)$ whose underlying scheme $\underline{X} = \operatorname{Spec} A$ is affine. Since $\mathcal{M}_X$ is coherent, we know that there exists an affine open cover $\{\underline{X}_i\}_{i \in I}$ as above, such that for each i the log scheme $(\underline{X}_i, \mathcal{M}_X|_{X_i})$ admits a (global) chart. Since the $\underline{X}_i$ are affine, such a chart induces the data of a finitely generated monoid P_i and a morphism

$$P_i \to A_i$$

from which the log structure $\mathcal{M}_X|_{X_i}$ can be recovered. As in the case of coherent sheaves, the natural question is then the following: can we glue these local charts together, in order to produce a global chart $P \to A$? If we could, then we would be able to say that the category of coherent log structures on $\underline{X} = \operatorname{Spec} A$ is equivalent to the category of finitely-generated monoids P equipped with a monoid morphism $P \to A$.

Unfortunately, it turns out that this is *not true*; there are perfectly reasonable examples of coherent affine log schemes which do not admit global charts. Consequently, there is no purely algebraic description of the category of log structures on a given scheme, even affine-locally.

In my personal opinion, this fact lies behind much of the confusion and discomfort caused by log structures. We are used to being able to reduce everything to a problem of algebra, at least locally. But with log structures, we are forced to confront sheaves for what they really are: the data of, for each open set, a monoid, and for each inclusion of open sets, a restriction map. Of course, we can always take smaller open sets until we find one which has a chart, but we are often not able to say how small these sets have to be. Therefore in doing log geometry, we are forced to work with open sets, and so to think a little more topologically and a little less algebraically.

Example 3.5.28. There are numerous examples of affine log schemes with no global chart. However, though it is easy to write down such a log scheme and convince oneself that none of the obvious ways of constructing a global chart work, actually proving that *no chart exists* can be quite tricky. Here we present an example first suggested to us by O. Overkamp, in which we were actually able to prove the non-existence of a global chart.

Let $\underline{X}$ be a smooth, connected affine variety and let $\underline{D} \subseteq \underline{X}$ be a smooth, connected divisor which is not linearly trivial, i.e. such that $\mathcal{O}_{\underline{X}}(\underline{D}) \ncong \mathcal{O}_{\underline{X}}$. Such examples do indeed exist; for instance, take $\underline{X}$ to be a smooth elliptic curve minus a point, and take $\underline{D}$ to be any point on $\underline{X}$.

Let $\mathcal{M}_X$ be the divisorial log structure on $\underline{X}$ with respect to $\underline{D}$ (this is coherent: see Example 3.5.26 above).

Proposition 3.5.29. There does not exist a global chart for $X = (\underline{X}, \mathcal{M}_X)$.

Proof. Suppose for a contradiction that such a chart exists. This means that there is a finitely generated monoid P and a strict morphism:

$$(\underline{X}, \mathcal{M}_X) \xrightarrow{f} \operatorname{Spec}(P \to \Bbbk[P]) =: (\underline{Y}, \mathcal{M}_Y)$$

Now, since P is finitely generated there exists a surjection $\mathbb{N}^r \to P$ for some $r \geq 0$, inducing a log morphism:

$$(\underline{Y}, \mathcal{M}_Y) := \operatorname{Spec}(P \to \Bbbk[P]) \xrightarrow{g} \operatorname{Spec}(\mathbb{N}^r \to \Bbbk[\mathbb{N}^r]) =: (\underline{Z}, \mathcal{M}_Z)$$

Here, of course, $(\underline{Z}, \mathcal{M}_Z)$ is just $\mathbb{A}^r$ equipped with the divisorial log structure corresponding to the co-ordinate hyperplanes. The log map g induces a diagram of abelian sheaves on Y:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_Y^* & \longrightarrow & g^*\mathcal{M}_Z^{\mathrm{gp}} & \longrightarrow & g^*\overline{\mathcal{M}}_Z^{\mathrm{gp}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_Y^* & \longrightarrow & \mathcal{M}_Y^{\mathrm{gp}} & \longrightarrow & \overline{\mathcal{M}}_Y^{\mathrm{gp}} & \longrightarrow & 0 \end{array}$$

Taking global sections of the ghost sheaves, we obtain a sequence of maps:

$$\mathbb{Z}^r = \Gamma(Z, \overline{\mathcal{M}}_Z^{\mathrm{gp}}) \xrightarrow{g^*} \Gamma(Y, g^*\overline{\mathcal{M}}_Z^{\mathrm{gp}}) \xrightarrow{g^\flat} \Gamma(Y, \overline{\mathcal{M}}_Y^{\mathrm{gp}})$$

We denote the composition by $\varphi \colon \mathbb{Z}^r \to \Gamma(Y, \overline{\mathcal{M}}_Y^{\mathrm{gp}})$. Given $v \in \mathbb{Z}^r$ there is a line bundle on $\underline{Y}$ associated to the global section $\varphi(v)$ of $\overline{\mathcal{M}}_Y^{\mathrm{gp}}$. By the above diagram, this line bundle must be trivial, since it is equal to the pull-back along g of the line bundle on $\underline{Z}$ associated to v, which is trivial since $\operatorname{Pic} \underline{Z} = 0$.

Now let us examine the corresponding diagram for the log map f

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X^* & \longrightarrow & f^*\mathcal{M}_Y^{\mathrm{gp}} & \longrightarrow & f^*\overline{\mathcal{M}}_Y^{\mathrm{gp}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & \mathcal{O}_X^* & \longrightarrow & \mathcal{M}_X^{\mathrm{gp}} & \longrightarrow & \overline{\mathcal{M}}_X^{\mathrm{gp}} & \longrightarrow & 0 \end{array}$$

where now the vertical maps are isomorphisms because f is strict. Taking global sections, we have

$$\Gamma(X, f^*\overline{\mathcal{M}}_Y^{\mathrm{gp}}) = \Gamma(X, \overline{\mathcal{M}}_X^{\mathrm{gp}}) = \mathbb{Z}$$

with canonical generator $1 \in \mathbb{Z}$ inducing the line bundle $\mathcal{O}_{\underline{X}}(\underline{D})$ on $\underline{X}$. We have maps

$$\mathbb{Z}^r \xrightarrow{\varphi} \Gamma(Y, \overline{\mathcal{M}}_Y^{\mathrm{gp}}) \xrightarrow{\psi} \Gamma(Y, f^*\overline{\mathcal{M}}_Y^{\mathrm{gp}}) = \mathbb{Z}$$

and we claim that the composition $\psi \circ \varphi$ is surjective. Once this is proven, we will immediately arrive at a contradiction. For suppose there exists a $v \in \mathbb{Z}^r$ such that $\psi \circ \varphi(v) = 1$. As discussed above, the line bundle on $\underline{X}$ associated to $1 \in \mathbb{Z}$ is $\mathcal{O}_{\underline{X}}(\underline{D})$. On the other hand, this is isomorphic to the pull-back along f of the line bundle on $\underline{Y}$ associated to $\varphi(v)$, and we saw before that this was trivial. Thus we obtain

$$\mathcal{O}_{\underline{X}}(\underline{D}) \cong \mathcal{O}_{\underline{X}}$$

which contradicts the assumption that $\underline{D}$ was not linearly trivial.

It remains to show that $\psi \circ \varphi$ is surjective; we start by examining the constructible sheaf $\overline{\mathcal{M}}_Y^{\mathrm{gp}}$. By the definition of the log structure $\mathcal{M}_Y$, there is a natural isomorphism of sheaves on $\underline{Y}$ [Gil09, Lemma 4.3.4]

$$\overline{\mathcal{M}}_Y = \underline{P}/\alpha^{-1}(\mathcal{O}_Y^*)$$

where α is the sheaf map $\underline{P} \to \mathcal{O}_Y$. Thus in particular we have a surjective morphism of constructible sheaves:

$$\underline{P} \twoheadrightarrow \overline{\mathcal{M}}_Y$$

Groupifying, we obtain a morphism $\underline{P}^{\mathrm{gp}} \to \overline{\mathcal{M}}_Y^{\mathrm{gp}}$ of constructible abelian sheaves; by examining the stalks, we see that this is still surjective. Pulling back along f, we obtain a morphism of sheaves on $\underline{X}$

$$\underline{P}^{\mathrm{gp}} \twoheadrightarrow f^{-1}\overline{\mathcal{M}}_Y^{\mathrm{gp}}$$

which is surjective by exactness of the pull-back functor. By [Gil09, Proposition 5.5.1] there is a natural isomorphism $f^{-1}\overline{\mathcal{M}}_Y^{\mathrm{gp}} = f^*\overline{\mathcal{M}}_Y^{\mathrm{gp}}$. (We do not have to distinguish between $f^{-1}(\overline{\mathcal{M}}_Y^{\mathrm{gp}})$ and $(f^{-1}\overline{\mathcal{M}}_Y)^{\mathrm{gp}}$ because groupification commutes with taking stalks.) Moreover by constructibility the induced morphism on global sections is also surjective, and so we obtain a composition of surjective maps

$$\mathbb{Z}^r \twoheadrightarrow P^{\mathrm{gp}} \twoheadrightarrow \Gamma(Y, f^*\overline{\mathcal{M}}_Y^{\mathrm{gp}})$$

which is precisely the morphism $\psi \circ \varphi$ considered earlier. Thus $\psi \circ \varphi$ is surjective, and this completes the proof. □

3.5.3. **Integral and saturated log schemes.**

3.5.3.1. *Integral monoids.* A monoid P is said to be *integral* if it satisfies the following cancellative property for $p, q, r \in P$:

$$p + r = q + r \Rightarrow p = q$$

This notion is roughly analogous to that of an integral domain for rings. Indeed, we have the following result.

Lemma 3.5.30. Let P be any monoid, and suppose that $\Bbbk[P]$ is an integral domain. Then P is integral.

Proof. We prove the contrapositive. Suppose that P is not integral, so we have $p, q, r \in P$ with $p \neq q$ and $p + r = q + r$. Then in $\Bbbk[P]$ we have

$$z^p \cdot z^r = z^q \cdot z^r \Rightarrow (z^p - z^q) \cdot z^r = 0$$

but $z^r \neq 0$ since $r \neq 0$ and $z^p - z^q \neq 0$ since $p \neq q$. Thus, $\Bbbk[P]$ is not an integral domain. □

We note that the converse of the above statement does not hold. There is also an alternative characterisation of integrality, which is often useful:

Lemma 3.5.31. A monoid P is integral if and only if the natural map $P \to P^{\mathrm{gp}}$ is injective.

Example 3.5.32. The natural numbers $\mathbb{N}$ are integral, as is every group.

Example 3.5.33. For an easy example of a non-integral monoid, consider the monoid $\mathbb{N}^{\mathrm{strange}}$ consisting of the natural numbers under *multiplication.* This is not integral, since for instance:

$$1 \cdot 0 = 2 \cdot 0$$

However, this is not really a good example because $\mathbb{N}^{\mathrm{strange}}$ is quite an odd monoid; for instance, it is not even finitely generated (there are infinitely many primes).

Example 3.5.34. For a more reasonable example involving a monoid which might actually appear in nature, consider:

$$P = \mathbb{N}^2/((1,0) = (1,1))$$

Consider the element $(0,1) \in P$. We have

$$(0,1) + (1,0) = (1,1) \qquad (0,0) + (1,0) = (1,0) = (1,1)$$

but we claim that $(0,1) \neq (0,0)$. To see this, we observe that P is the coequaliser of the following diagram:

$$\mathbb{N} \underset{(1,1)}{\overset{(1,0)}{\rightrightarrows}} \mathbb{N}^2$$

Consider the following monoid morphism:

$$\begin{aligned} \mathbb{N}^2 &\to \Bbbk[x,y]/(x - xy) = R \\ (a,b) &\to x^a y^b \end{aligned}$$

This coequalises the two maps above, hence there is a unique factorisation through P:

$$\mathbb{N}^2 \to P \to \Bbbk[x,y]/(x - xy) = R$$

We note that $(0,1) \in \mathbb{N}^2$ is sent to $y \in R$ which is not equal to the monoid identity $1 \in R$. Hence by the above factorisation we must have that $(0,1) \neq (0,0)$ in P, as claimed.

3.5.3.2. *Integralisation of monoids.* Every monoid has a minimal associated integral monoid, called the *integralisation*. It is defined as the image of the map:

$$P \to P^{\mathrm{gp}}$$

Any morphism from P to an integral monoid must factor uniquely through P^{int}. By the above construction, the map $P \to P^{\mathrm{int}}$ is always surjective, and is an isomorphism if and only if P is integral. Note that we always have $(P^{\mathrm{int}})^{\mathrm{gp}} = P^{\mathrm{gp}}$.

Example 3.5.35. Consider Example 3.5.34 above. First we need to calculate P^{gp}. Looking at the coequaliser diagram and groupifying, we obtain a diagram:

$$\begin{array}{ccccccc} \mathbb{N} & \underset{(1,1)}{\overset{(1,0)}{\rightrightarrows}} & \mathbb{N}^2 & \longrightarrow & P & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{Z} & \underset{(1,1)}{\overset{(1,0)}{\rightrightarrows}} & \mathbb{Z}^2 & \xrightarrow[(a,b)\mapsto a]{} & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

The map $P \to \mathbb{Z}$ is obtained via the universal property of the coequaliser, but a quick diagram chase shows that in fact this map is the groupification map $P \to P^{\mathrm{gp}}$. Thus we see that

$$P^{\mathrm{int}} = \mathrm{Im}(P \to P^{\mathrm{gp}}) = \mathbb{N}$$

with map $P \to \mathbb{N}$ given by $(a, b) \mapsto a$. Note that this map is not injective, which is what we expect since P is not integral.

3.5.3.3. *Fine log schemes.* A log scheme $(X, \mathcal{M}_X)$ is called *fine* if it is coherent (which, remember, means it admits local charts given by finitely generated monoids P) and if moreover the monoids P can chosen to be integral.

Example 3.5.36. The divisorial log structure associated to a smooth pair $(\underline{X}, \underline{D})$ (see Example 3.5.26) is fine. More generally, the divisorial log structure associated to a pair $(\underline{X}, \underline{D})$ with $\underline{X}$ smooth and $\underline{D}$ normal crossings, is fine.

Example 3.5.37. The log structure on a point constructed in Example 3.5.7 is fine if and only if Q is integral.

3.5.3.4. *Integralisation of log schemes.* Fine log schemes form a full subcategory of coherent log schemes, denoted:

$$\mathbf{LogSch}^{\mathbf{f}} \hookrightarrow \mathbf{LogSch}^{\mathbf{coh}}$$

This inclusion admits a left adjoint, i.e. for every coherent log scheme there is a unique minimal fine log scheme associated to it, called its *integralisation.*

We will describe this construction on the level of charts; the general construction follows from this one by descent. Suppose then that we have a coherent log scheme $X = (\underline{X}, \mathcal{M}_X)$ which admits a global chart, i.e. a strict morphism:

$$X \to \mathrm{Spec}\,(P \to \Bbbk[P])$$

Since our log scheme X is not necessarily fine, P is not necessarily integral. The morphism $P \to P^{\mathrm{int}}$ dualises to give a morphism of log schemes

$$\mathrm{Spec}\,(P^{\mathrm{int}} \to \Bbbk[P^{\mathrm{int}}]) \to \mathrm{Spec}\,(P \to \Bbbk[P])$$

and we define the integralisation of X to the fibre product in the category of log schemes:

$$\begin{array}{ccc} X^{\text{int}} & \longrightarrow & \operatorname{Spec}(P^{\text{int}} \to \Bbbk[P^{\text{int}}]) \\ \downarrow & \square & \downarrow \\ X & \longrightarrow & \operatorname{Spec}(P \to \Bbbk[P]) \end{array}$$

Since strictness is a property of morphisms which is stable under base change (this follows easily by the construction of fibre products in $\mathbf{LogSch^{coh}}$) the upper horizontal map is a strict morphism, hence is a chart. This shows that X^{int} is a fine log scheme.

Note that in particular we have for the underlying schemes:

$$\underline{X}^{\text{int}} = \underline{X} \times_{\operatorname{Spec} \Bbbk[P]} \operatorname{Spec} \Bbbk[P^{\text{int}}]$$

So the process of integralisation can actually change the underlying scheme! What does this look like? We recall that the map $P \to P^{\text{int}}$ is surjective by construction, so the map

$$\operatorname{Spec} \Bbbk[P^{\text{int}}] \to \operatorname{Spec} \Bbbk[P]$$

is a closed embedding. Hence

$$\underline{X}^{\text{int}} \to \underline{X}$$

is also a closed embedding. In our study of relative Gromov–Witten invariants, we will see examples where this map appears as the inclusion of the main component of a moduli space.

Remark 3.5.38. Here we will try to convince the reader that this modification of the underlying scheme is in fact necessary. Viewing the chart above as a map

$$\underline{P} \to \mathcal{M}_X$$

it is tempting to try to keep $\underline{X}$ the same and simply replace $\underline{P}$ by $\underline{P}^{\text{int}}$. The obvious thing to try to do is construct $\mathcal{M}_X^{\text{int}}$ as the following pushout in the category of monoid sheaves on $\underline{X}$:

$$\begin{array}{ccc} \underline{P} & \longrightarrow & \mathcal{M}_X \\ \downarrow & \lrcorner & \downarrow \\ \underline{P}^{\text{int}} & \longrightarrow & \mathcal{M}_X^{\text{int}} \end{array}$$

The problem is that there is no natural morphism $\mathcal{M}_X^{\text{int}} \to \mathcal{O}_X$. There is, however a morphism from $\mathcal{M}_X^{\text{int}}$ to a certain quotient of $\mathcal{O}_X$, which is precisely the structure sheaf of the closed subscheme $\underline{X}^{\text{int}}$ constructed above.

3.5.3.5. *Fibre products in* $\mathbf{LogSch}^{\mathbf{f}}$. In Definition-Lemma 3.5.21 we demonstrated how to form a fibre product in the category **LogSch** of all log schemes, and saw that the underlying scheme of the fibre product was equal to the fibre product of the underlying schemes.

It is easy to see that a fibre product of coherent log schemes will still be coherent; if we have a diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

and charts

$$P_X \to \Gamma(X, \mathcal{M}_X) \qquad P_Y \to \Gamma(Y, \mathcal{M}_Y) \qquad P_Z \to \Gamma(Z, \mathcal{M}_Z)$$

then (after possibly passing to an open cover; see [Kat89, Definition 2.9.2]) we can find compatible morphisms of monoids

$$P_Z \to P_X \qquad P_Z \to P_Y$$

inducing the maps on monoid sheaves (such a morphism is called a *chart* for the log map). We then get an induced map:

$$P_X \oplus_{P_z} P_Y \to \Gamma(X, \mathcal{M}_X) \oplus_{\Gamma(Z, \mathcal{M}_Z)} \Gamma(Y, \mathcal{M}_Y) \to \Gamma(X \times_Z Y, \mathcal{M}_{X \times_Z Y})$$

and it is straightforward to show that this gives a chart for $X \times_Z Y$. Thus, the fibre product in the category $\mathbf{LogSch}^{\mathbf{coh}}$ of coherent log schemes exists and agrees with the fibre product in **LogSch**.

On the other hand, there is no reason to suppose that the monoid $P_X \oplus_{P_Z} P_Y$ is integral, even if P_X, P_Y and P_Z are (see Example 3.5.40 below). Thus, a fibre product of fine log schemes is not necessarily fine. Nevertheless, we have the following result.

Definition-Lemma 3.5.39. Fibre products exist in $\mathbf{LogSch}^{\mathbf{f}}$.

Proof. The construction is as follows: first take the fibre product in $\mathbf{LogSch}^{\mathbf{coh}}$ and then integralise the result. The fact that this produces a fibre product in $\mathbf{LogSch}^{\mathbf{f}}$ follows from formal nonsense about adjoint functors. □

So fibre products exist in $\mathbf{LogSch}^{\mathbf{f}}$, but the underlying scheme of the fibre product is not in general equal to the fibre product of the underlying schemes (rather, it is a closed subscheme thereof). This observation will turn out to be important for us, since the log structures on the source curve and the base of a log stable map are required to be fine.

Example 3.5.40. Here we provide an example where a fibre product of fine log schemes in the category $\mathbf{LogSch^{coh}}$ can produce a log scheme which is not fine (I am grateful to M. Gross for showing me this example). Consider the following diagram of ordinary schemes

$$\begin{array}{ccc} & & \underline{X} = \mathbb{A}^1_t \\ & & \downarrow \\ \underline{Y} = \mathrm{Bl}_{(0,0)}\,\mathbb{A}^2_{xy} & \longrightarrow & \underline{Z} = \mathbb{A}^2_{xy} \end{array}$$

where $\underline{X} \to \underline{Z}$ is given by $x \mapsto t, y \mapsto t$ (i.e. it is the inclusion of the diagonal hyperplane) and $\underline{Y} \to \underline{Z}$ is the blow-up map. We equip these schemes with the following divisorial log structures:

- $\mathcal{M}_X$ the divisorial log structure corresponding to the origin;
- $\mathcal{M}_Y$ the divisorial log structure corresponding to the toric boundary;
- $\mathcal{M}_Z$ the divisorial log structure corresponding to the co-ordinate hyperplanes.

Note that since these log structures are all divisorial, their structure maps are inclusions, and hence a log enhancement of a morphism between the underlying schemes is unique if it exists. Moreover it is straightforward to check that for the maps in the above diagram, such an enhancement does indeed exist, so that we obtain a diagram of log schemes:

$$\begin{array}{ccc} & & X = (\underline{X}, \mathcal{M}_X) \\ & & \downarrow \\ Y = (\underline{Y}, \mathcal{M}_Y) & \longrightarrow & Z = (\underline{Z}, \mathcal{M}_Z) \end{array}$$

All of these log structures are fine, and we can write down charts explicitly. Those for X and Z exist globally and are easy to understand; they are given by

$$P_X = \mathbb{N} \to \Bbbk[t]$$
$$n \mapsto t^n$$

and:

$$P_Z = \mathbb{N}^2 \to \Bbbk[x, y]$$
$$(a, b) \mapsto x^a y^b$$

To obtain a chart for Y, we must work locally. Let us restrict to an open neighbourhood in $\underline{Y}$ which contains the intersection of the exceptional divisor with (the proper transform of) $\{y = 0\}$, and which does not touch (the proper transform of) $\{x = 0\}$. Let A be the co-ordinate ring of this affine open set and let z be the local equation cutting out the exceptional divisor.

Then a chart is given by:

$$P_Y = \mathbb{N}^2 \to A$$
$$(c,d) \mapsto z^c y^d$$

In addition to the log schemes X, Y and Z having charts, the morphisms $X \to Z$ and $Y \to Z$ also admit charts, given by

$$P_Z = \mathbb{N}^2 \to \mathbb{N} = P_X$$
$$(a,b) \mapsto a+b$$

and

$$P_Z = \mathbb{N}^2 \to \mathbb{N}^2 = P_Y$$
$$(a,b) \mapsto (a+b,b)$$

(see [Kat89, Definition 2.9] for the definition of a chart for a morphism of log schemes). As discussed at the start of this section, a chart for $X \times_Z Y$ is then given by the fibred coproduct:

$$P_X \oplus_{P_Z} P_Y = \mathbb{N} \oplus_{\mathbb{N}^2} \mathbb{N}^2$$

We claim that this monoid is not integral. Note that

$$(1,(0,0)) + (0,(0,1)) = (0,(1,0)) + (0,(0,1)) = (0,(1,1)) = (1,(0,0))$$

while on the other hand $(0,(0,1)) \neq 0$, as can be seen by a similar argument to the one given in Example 3.5.34. So the monoid is not integral.

In this example, the fibre product of the underlying schemes is the closed subscheme of $\underline{Y}$ consisting of the union of the exceptional divisor with (the proper transform of) the diagonal hyperplane. This is the scheme underlying the fibre product in $\mathbf{LogSch}^{\mathbf{coh}}$. On the other hand, the scheme underlying the fibre product in $\mathbf{LogSch}^{\mathbf{f}}$ consists only of the exceptional divisor.

3.5.3.6. *Saturated monoids.* A monoid P is said to be *saturated* if it is integral and if, for all $p \in P^{\mathrm{gp}}$ and $n \in \mathbb{N}$ with $n \geq 1$, the following condition holds:

$$n \cdot p \in P \Rightarrow p \in P$$

Example 3.5.41. $\mathbb{N}$ is saturated, as is any group.

Example 3.5.42. Let $P = \mathbb{N} \setminus \{1\}$ with addition as the binary operation. This is a monoid (a submonoid of $\mathbb{N}$, indeed) and it is integral, but it is not saturated: we have $1 = 3 - 2 \in P^{\mathrm{gp}}$ with $2 \cdot 1 \in P$, but $1 \notin P$.

3.5.3.7. *Saturation of monoids.* Every monoid P has a unique minimal associated saturated monoid, called the *saturation.* It is defined as:

$$P^{\mathrm{sat}} = \{p \in P^{\mathrm{gp}} \colon n \cdot p \in P \text{ for some } n \geq 1\}$$

There is a natural inclusion $P \to P^{\mathrm{sat}}$ satisfying the obvious universal property.

Example 3.5.43. Consider Example 3.5.42 from above. The saturation P^{sat} is equal to $\mathbb{N}$, with $P \to P^{\mathrm{sat}}$ the natural inclusion.

3.5.3.8. *Fine and saturated (fs) log schemes.* A log scheme $X = (\underline{X}, \mathcal{M}_X)$ is said to be *fine and saturated* (or *fs* for short) if it is fine and if it admits local charts which are given by saturated monoids.

Example 3.5.44. $X = \mathbb{A}^1$ with the divisorial log structure induced by the origin is saturated. This is the log scheme $\operatorname{Spec}(\mathbb{N} \to \Bbbk[\mathbb{N}])$.

Example 3.5.45. Let $P = \mathbb{N} \setminus \{1\}$ be the monoid from Example 3.5.42 and let $X = \operatorname{Spec}(P \to \Bbbk[P])$. Then this log scheme is not saturated. To see what the underlying scheme is, note that P is generated by $a = 2$ and $b = 3$, subject to the relation $3a = 2b$. Hence

$$\Bbbk[P] = \Bbbk[x, y]/(x^3 - y^2)$$

and so $\underline{X}$ is the cuspidal cubic. The log structure is the divisorial log structure with respect to the singular point.

3.5.3.9. *Saturation of fine log schemes.* Similar to before, fs log schemes form a full subcategory inside fine log schemes

$$\mathbf{LogSch}^{\mathbf{fs}} \hookrightarrow \mathbf{LogSch}^{\mathbf{f}}$$

and as before this inclusion admits a left adjoint, called *saturation.* Given a fine log scheme with chart

$$X \to \operatorname{Spec}(P \to \Bbbk[P])$$

the saturation is defined as the fibre product

$$\begin{array}{ccc} X^{\text{sat}} & \longrightarrow & \operatorname{Spec}(P^{\text{sat}} \to \Bbbk[P^{\text{sat}}]) \\ \downarrow & \square & \downarrow \\ X & \longrightarrow & \operatorname{Spec}(P \to \Bbbk[P]) \end{array}$$

in the category $\mathbf{LogSch^{coh}}$. As before, it is immediate to check that the upper horizontal arrow is a chart, and hence that X^{sat} is saturated.

In particular we see that, as with integralisation, saturation can also change the underlying scheme structure. In this case the change which occurs is akin to the normalisation; the map $\underline{X}^{\text{sat}} \to \underline{X}$ is always finite.

Example 3.5.46. Consider again $P = \mathbb{N} \setminus \{1\}$ and $X = \operatorname{Spec}(P \to \Bbbk[P])$. We saw before that $\underline{X} = V(x^3 - y^2) \subseteq \mathbb{A}^2_{xy}$ and also that $P^{\text{sat}} = \mathbb{N}$. It follows that $\underline{X}^{\text{sat}} = \mathbb{A}^1_t$ and that the saturation morphism is given by:

$$\begin{aligned} \underline{X}^{\text{sat}} = \mathbb{A}^1_t &\to \underline{X} = V(x^3 - y^2) \\ t^2 &\leftarrow\!\shortmid x \\ t^3 &\leftarrow\!\shortmid y \end{aligned}$$

Thus we see that in this case the saturation is none other than the normalisation.

3.5.3.10. *Fibre products in* $\mathbf{LogSch^{fs}}$. As the reader has probably guessed by now, if one considers a diagram of fs log schemes

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

and takes the fibre product in the category $\mathbf{LogSch^{f}}$ of fine log schemes, it is not in general true that the resulting log scheme is saturated (see Example 3.5.47 below).

Thus, in much the same way as we did in §3.5.3.5, we form the fibre product in $\mathbf{LogSch^{fs}}$ by first taking the fibre product in $\mathbf{LogSch^{f}}$ and then saturating. As already noted, this will entail a further modification of the underlying scheme.

Example 3.5.47. Let $X = Y = Z = \mathbb{A}^1_t$ with divisorial log structure corresponding to the origin. Consider the maps $X \to Z$, $Y \to Z$ both given by $t \mapsto t^2$. The induced chart for the fibre product in $\mathbf{LogSch^{coh}}$ is given by

$$Q = \mathbb{N} \oplus_{\mathbb{N}} \mathbb{N}$$

where both maps $\mathbb{N} \to \mathbb{N}$ are given by $1 \mapsto 2$. This monoid is integral, and hence the fibre product in $\mathbf{LogSch}^{\mathbf{coh}}$ agrees with the fibre product in $\mathbf{LogSch}^{\mathbf{f}}$ and has chart given by Q. On the other hand, Q is not saturated, since

$$2 \cdot (1,-1) = (2,-2) = (0,2) + (0,-2) = (0,0) \in Q$$

while $(1,-1) \notin Q$.

3.5.3.11. *Conclusion.* How is all of this relevant for us? In our study of log Gromov–Witten theory, we encounter examples of moduli spaces with a logarithmic structure which is neither integral nor saturated. Integralising then singles out the main component of the moduli space, while saturating (log) desingularises that component. See §3.3.2 for details.

CHAPTER 4

The fundamental solution matrix and relative stable maps

The following chapter originally appeared as [Nab18].

Abstract: Givental's Lagrangian cone $\mathcal{L}_X$ is a Lagrangian submanifold of a symplectic vector space which encodes the genus-zero Gromov–Witten invariants of X. Building on work of Braverman, Coates has obtained the Lagrangian cone as the push-forward of a certain class on the moduli space of stable maps to $X \times \mathbb{P}^1$. This provides a conceptual description for an otherwise mysterious change of variables called the dilaton shift.

In this chapter we recast this construction in its natural context, namely the moduli space of stable maps to $X \times \mathbb{P}^1$ relative the divisor $X \times \infty$. We find that the resulting push-forward is another familiar object, namely the transform of the Lagrangian cone under the action of the fundamental solution matrix. This hints at a generalisation of Givental's quantisation formalism to the setting of relative invariants. Finally, we use a hidden polynomiality property implied by our construction to obtain a sequence of universal relations for the Gromov–Witten invariants, as well as new proofs of several foundational results concerning both the Lagrangian cone and the fundamental solution matrix.

4.1. Introduction

4.1.1. **Gromov–Witten theory**. The Gromov–Witten invariants of a smooth projective variety X are defined as certain intersection numbers on moduli spaces of stable maps to X. They can be thought of as counting curves of specified genus and degree passing through specified subvarieties of X. Their intrinsic interest aside, Gromov–Witten invariants have connections to numerous other areas of mathematics, from representation theory to symplectic topology. In algebraic geometry they have been used in the proofs of classification theorems, as a tool for distinguishing non-deformation-equivalent varieties.

Many results in Gromov–Witten theory are expressed most cleanly via generating functions, that is, formal functions (usually polynomials or power series) whose coefficients are given by Gromov–Witten invariants. Oftentimes, a simple identity involving generating functions is all that is needed to express a relationship which, on the level of indvidual invariants, is extremely complicated. There is an underlying reason for this: Gromov–Witten theory has deep connections to theoretical physics, through which the aforementioned generating functions appear as the "partition functions" of physical theories. This circle of ideas has been extremely influential for the development of the subject, with the first major result in this direction being the celebrated Mirror Theorem [CdlOGP91, Giv96, Giv98].

4.1.2. **Quantisation formalism**. In keeping with this spirit, A. Givental describes in [Giv01a] a *quantisation formalism* for Gromov–Witten invariants. In the genus-zero setting (when no "quantisation" is actually required), this amounts to encoding the Gromov–Witten invariants of X in a *Lagrangian cone*

$$\mathcal{L}_X \subseteq \mathcal{H}$$

inside a certain symplectic vector space $\mathcal{H}$, now called the *Givental space*. The data of the cone $\mathcal{L}_X$ is equivalent to the data of the generating functions discussed earlier, but it turns out to be a good idea to treat $\mathcal{L}_X$ as a geometric object in its own right; many statements in Gromov–Witten theory can then be translated into statements about how $\mathcal{L}_X$ transforms under certain symplectomorphisms of $\mathcal{H}$.

The benefits of this quantisation formalism are twofold. From a theoretical viewpoint, it can be used to make rigorous sense of a number of deep predictions coming from physics. On the other hand, from a practical point of view, it has proven to be an extremely versatile framework in which to formulate and prove statements about Gromov–Witten invariants. Indeed, there are many results in Gromov–Witten theory which would be difficult to even state without the quantisation formalism: examples include the quantum Riemann–Roch formula [CG07], the crepant transformation conjecture [CIJ14], the Virasoro conjecture and various versions of the "genus zero implies higher genus" principle [Giv01b]

4.1.3. **Push-forwards from graph spaces**. Building on work of Braverman [Bra04], T. Coates shows in [Coa08] that $\mathcal{L}_X$ can be obtained as a ($\mathbb{C}^*$-localised) push-forward from the moduli space of stable maps to $X \times \mathbb{P}^1$ (usually called the *graph space*). This is motivated by Givental's heuristic description of $\mathcal{H}$ as the S^1-equivariant cohomology of the loop space of X [Giv95], and gives a natural geometric interpretation for a mysterious change of variables, called the "dilaton shift", which is essential to the quantisation formalism.

Coates' construction requires restricting to a certain open substack of the moduli space of stable maps to $X \times \mathbb{P}^1$, before localising to a proper fixed locus (with respect to the natural $\mathbb{C}^*$-action on the moduli space) in order to push forward. With hindsight, this is really the push-forward from *one* of the $\mathbb{C}^*$-fixed loci in the moduli space of *relative* stable maps to the pair $(X \times \mathbb{P}^1, X \times \infty)$.

A natural question to ask is then: what happens if we sum over *all* the fixed loci? In this chapter we provide the answer (see Proposition 4.2.4): the result is the transform of the Lagrangian cone under the action of the fundamental solution matrix. The main tools used in the proof are the relative virtual localisation formula [GV05, Theorem 3.6], a virtual push-forward theorem for relative stable maps to the non-rigid target [Gat03a, Theorem 5.2.7] and a comparison lemma for psi classes, which we prove in §4.3.2.

Because we are now summing over all fixed loci, we know that the resulting class must actually belong to the *non-localised* equivariant cohomology. In practice, this means the following: we push-forward and obtain a class which, a priori, looks like a rational function in z; however we know that, after performing suitable cancellations, we must end up with a polynomial (here z denotes the $\mathbb{C}^*$-equivariant parameter). We use this observation to give new and simple proofs of a number of foundational results belonging to the quantisation formalism theory.

4.1.4. **Future directions**. This construction provides a hint as to how one might obtain a quantisation formalism for relative (or logarithmic) Gromov–Witten invariants; see Remark 4.2.3. This was in fact the original motivation for this work.

4.1.5. **User's guide**. Readers familiar with the quantisation formalism may skip straight to §4.2.5 where we give the statement of the main result. For the uninitiated, we provide in §§4.2.1-4.2.4 a brief introduction to the Lagrangian cone and relative Gromov–Witten theory. The proof of the main result is given in §4.3; this is mostly a computation, with the only geometric content being a lemma on psi classes which we prove in §4.3.2. Finally in §4.4 we provide examples of how the "hidden polynomiality" implied by our construction can be used to obtain universal relations for the Gromov–Witten invariants, as well as new proofs of a number of standard results concerning the Lagrangian cone and the fundamental solution matrix.

4.1.6. **Acknowledgements**. I owe a great deal of thanks to Tom Coates, for first suggesting this project, for patiently explaining the quantisation formalism to me and for pointing out some of the applications presented in §4.4. I would also like to thank Pierrick Bousseau, Elana Kalashnikov and Mark Shoemaker for useful discussions.

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4.2. Background and statement of the main result

4.2.1. **Givental space**. The Lagrangian cone $\mathcal{L}_X$ is a geometric object which encodes all the genus-zero Gromov–Witten invariants of X. It can be viewed as the graph of a certain generating function for these invariants. This generating function must keep track, through its formal variables, of both the cohomological insertions γ_i and the exponents k_i of the classes ψ_i. We begin by defining a vector space $\mathcal{H}$ whose co-ordinates will give precisely these formal variables; the Lagrangian cone will then be a submanifold of $\mathcal{H}$.

We set $\mathrm{H}^*(X) = \mathrm{H}^*(X;\Lambda)$ where Λ is some (unspecified) field of characteristic zero; for the moment it is safe to take $\Lambda = \mathbb{C}$, but later we will need to consider larger fields. We assume (for notational simplicity) that X has only even cohomology, and choose a homogeneous basis $\varphi_0, \ldots, \varphi_N$ such that $\varphi_0 = \mathbb{1}_X$ is the unit element. We let $\varphi^0, \ldots, \varphi^N$ denote the dual basis with respect to the Poincaré pairing $(\cdot,\cdot)$, so that:

$$(\varphi_\alpha, \varphi^\beta) = \delta_\alpha^\beta$$

The *Givental space* $\mathcal{H}$ is a certain infinite-dimensional symplectic vector space (over Λ) associated to X. It is defined as the space of formal Laurent series in a single variable z^{-1} with coefficients in $\mathrm{H}^*(X)$:

$$\mathcal{H} := \mathrm{H}^*(X)[z, z^{-1}]\!] = \left\{ \sum_{-\infty \leq k \leq m} q_k z^k : q_k \in \mathrm{H}^*(X) \right\}$$

The notation above is meant to indicate that each series has only finitely many positive powers of z, but can have infinitely many negative powers. The powers of z^{-1} will keep track of the exponents of the psi classes.

There is a symplectic form Ω on $\mathcal{H}$ defined as follows

$$\begin{aligned} \Omega\colon \mathcal{H} \times \mathcal{H} &\to \Lambda \\ (f(z), g(z)) &\mapsto \mathrm{Res}_{z=0}(f(-z), g(z))\,\mathrm{d}z \end{aligned}$$

where $(f(-z), g(z))$ is the Poincaré pairing (extended linearly from $\mathrm{H}^*(X)$ to $\mathcal{H}$), and $\mathrm{Res}_{z=0}$ simply means that we take the coefficient of z^{-1} in the resulting Laurent series. A straightforward computation verifies that Ω is indeed a symplectic form.

Example 4.2.1. Take $X = \mathrm{pt}$ so that $\mathrm{H}^*(X) = \Lambda$. Then $\mathcal{H} = \Lambda[z, z^{-1}]\!]$ and Ω is given by:

$$\Omega\left(\sum_k a_k z^k, \sum_l b_l z^l\right) = \mathrm{Res}_{z=0}\left(\sum_k \sum_l (-1)^k a_k b_l z^{k+l}\right) = \sum_{k+l=-1} (-1)^k a_k b_l$$

Notice that this sum is finite since the terms which appear must have either k or l non-negative, and there are only finitely many such values for which a_k and b_l are both non-zero.

Thus $(\mathcal{H}, \Omega)$ is an infinite-dimensional symplectic vector space. We will now write down Darboux co-ordinates. It is clear that the following defines a basis for $\mathcal{H}$:

$$A^k_\alpha := \varphi_\alpha z^k \qquad k \geq 0, \alpha = 0, \ldots, N$$
$$B^\gamma_l := \varphi^\gamma (-z)^{-1-l} \qquad l \geq 0, \gamma = 0, \ldots, N$$

It is also easy to see that these give Darboux co-ordinates, i.e. that we have:

$$\Omega(A^k_\alpha, A^{k'}_{\alpha'}) = 0 \qquad \Omega(B^\gamma_l, B^{\gamma'}_{l'}) = 0 \qquad \Omega(A^k_\alpha, B^\gamma_l) = -\delta^\gamma_\alpha \delta^k_l$$

Using these canonical co-ordinates we can define linear subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ to be the spans, respectively, of the A^k_α and B^γ_l inside $\mathcal{H}$:

$$\mathcal{H}_+ := \mathrm{H}^*(X)[z] = \left\{\sum_{k\geq 0} q^\alpha_k \varphi_\alpha z^k : q^\alpha_k \in \Lambda\right\}$$
$$\mathcal{H}_- := z^{-1}\,\mathrm{H}^*(X)[\![z^{-1}]\!] = \left\{\sum_{l\geq 0} p^l_\gamma \varphi^\gamma (-z)^{-1-l} : p^l_\gamma \in \Lambda\right\}$$

Here, and in what follows, we adopt the Einstein summation convention when dealing with Greek letters, i.e. when summing over cohomology classes φ_α and φ^γ. It is clear that both $\mathcal{H}_+$ and $\mathcal{H}_-$ are Lagrangian subspaces, in the sense that:

$$\mathcal{H}^\perp_\pm = \left\{v \in \mathcal{H} \;\middle|\; \Omega(v, w) = 0 \text{ for all } w \in \mathcal{H}_\pm\right\} = \mathcal{H}_\pm$$

Thus we think of $\mathcal{H}_+$ and $\mathcal{H}_-$ as being "half-dimensional" or "semi-infinite" (since in the finite-dimensional setting a Lagrangian subspace is always half-dimensional). Furthermore this splitting gives an identification of symplectic vector spaces

$$\mathcal{H} = \mathrm{T}^*\,\mathcal{H}_+$$

which means that $\mathcal{H}_-$ gets identified with the cotangent fibre; in terms of the co-ordinates q_k^α, p_γ^l above, the identification is:

$$p_\alpha^k = \frac{\partial}{\partial q_k^\alpha}$$

4.2.2. **Lagrangian cone.** We are now in a position to construct the Lagrangian cone $\mathcal{L}_X$. A standard object in Gromov–Witten theory is the *genus-zero descendant potential*, which is a formal generating function for the genus-zero Gromov–Witten invariants:

$$\mathcal{F}_X^0(\mathbf{t}(z)) = \sum_{\beta,n} \frac{Q^\beta}{n!} \langle \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_n) \rangle_{0,n,\beta}^X$$

Let us explain the notation above. The sum is over all curve classes $\beta \in \mathrm{H}_2^+(X)$ and non-negative integers $n \geq 0$. The variable Q is a formal variable, called the *Novikov variable*, which keeps track of the curve class. We make sense of this by taking the ground field Λ to be the *Novikov field*:

$$\Lambda = \mathbb{C}((\mathrm{H}_2^+(X)))$$

Remember that we defined $\mathrm{H}^*(X) = \mathrm{H}^*(X; \Lambda)$ for some unspecified field Λ; from now on we take Λ to be the Novikov field. The parameter $\mathbf{t}(z)$ of the generating function is a formal power series with coefficients in $\mathrm{H}^*(X)$

$$\begin{aligned} \mathbf{t}(z) &= \sum_{k \geq 0} t_k z^k && t_k \in \mathrm{H}^*(X) \\ &= \sum_{k \geq 0} t_k^\alpha \varphi_\alpha z^k && t_k^\alpha \in \Lambda \end{aligned}$$

so that the correlators above are interpreted as

$$\begin{aligned} \langle \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_n) \rangle_{0,n,\beta}^X &:= \left\langle \sum_{k_1 \geq 0} t_{k_1}^{\alpha_1} \varphi_{\alpha_1} \psi_1^{k_1}, \ldots, \sum_{k_n \geq 0} t_{k_n}^{\alpha_n} \varphi_{\alpha_n} \psi_n^{k_n} \right\rangle_{0,n,\beta}^X \\ &= \sum_{k_1, \ldots, k_n \geq 0} t_{k_1}^{\alpha_1} \cdots t_{k_n}^{\alpha_n} \langle \varphi_{\alpha_1} \psi_1^{k_1}, \ldots, \varphi_{\alpha_n} \psi_n^{k_n} \rangle_{0,n,\beta}^X \end{aligned}$$

(remember that we are using the Einstein summation convention for the Greek letters). Thus we may rewrite $\mathcal{F}_X^0$ in a more transparent (though less convenient) form as:

$$\mathcal{F}_X^0(\mathbf{t}(z)) = \sum_{\beta,n} \frac{Q^\beta}{n!} \sum_{k_1, \ldots, k_n \geq 0} t_{k_1}^{\alpha_1} \cdots t_{k_n}^{\alpha_n} \cdot \langle \varphi_{\alpha_1} \psi_1^{k_1}, \ldots, \varphi_{\alpha_n} \psi_n^{k_n} \rangle_{0,n,\beta}^X$$

We view this as a formal power series in the variables t_k^α for $k \geq 0$ and $\alpha = 0, \ldots, N$. Notice that these co-ordinates look very similar to the co-ordinates q_k^α for $\mathcal{H}_+$ defined in §4.2.1; indeed they are related by the following change of variables

$$\mathbf{q}(z) = \mathbf{t}(z) - z\mathbb{1}_X$$

called the *dilaton shift*. In concrete terms this means that $q_k^\alpha = t_k^\alpha$ unless $(k, \alpha) = (1, 0)$, in which case $q_1^0 = t_1^0 - 1$. Under this change of variables, we can view $\mathcal{F}_X^0$ as a function

$$\mathcal{F}_X^0 \colon \mathcal{H}_+ \to \Lambda$$

and hence the derivative $\mathrm{d}\mathcal{F}_X^0$ defines a section of the cotangent bundle $\mathrm{T}^* \mathcal{H}_+$. The Lagrangian cone is defined as the graph of this section:

$$\mathcal{L}_X := \left\{ (\mathbf{q}(z), \mathbf{p}(z)) \in \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \;\middle|\; \mathbf{p}(z) = \mathrm{d}\mathcal{F}_X^0(\mathbf{q}(z)) \right\}$$

Thus for every point $\mathbf{q}(z) \in \mathcal{H}_+$ there is a unique point of $\mathcal{L}_X$ lying over $\mathbf{q}(z)$. In concrete terms, this is:

$$\begin{aligned} \mathcal{L}_X|_{\mathbf{q}(z)} &= (\mathbf{t}(z) - z\mathbb{1}_X) + \sum_{\beta,n} \frac{Q^\beta}{n!} \sum_{l \geq 0} \left\langle \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_n), \varphi_\gamma \psi_{n+1}^l \right\rangle_{0,n+1,\beta}^X \cdot \varphi^\gamma (-z)^{-l-1} \\ &= (\mathbf{t}(z) - z\mathbb{1}_X) + \sum_{\beta,n} \frac{Q^\beta}{n!} \left\langle \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_n), \left(\frac{\varphi_\gamma}{-z - \psi_{n+1}} \right) \right\rangle_{0,n+1,\beta}^X \cdot \varphi^\gamma \end{aligned}$$

The first term $\mathbf{t}(z) - z\mathbb{1}_X = \mathbf{q}(z)$ specifies the point in the base, while the remaining terms specify the point in the fibre. The meaning of the fractional insertion in the second line is that it should be expanded as a power series in z^{-1}, the result of which is precisely the expression on the first line.

As it has been presented, divorced from its origins in physics, $\mathcal{L}_X$ may come across as a mysterious object. Working with it takes some getting used to, but the eventual payoff is significant, and it is now recognised as a fundamental tool in Gromov–Witten theory. To give just a taste of this, we state a few basic facts about the Lagrangian cone.

Theorem 4.2.2 ([CG07, Proposition 1]). The following basic properties hold:

- $\mathcal{L}_X$ is a cone (it is preserved under scalar multiplication by elements of Λ);
- for $f \in \mathcal{L}_X$, we have $(\mathrm{T}_f \mathcal{L}_X) \cap \mathcal{L}_X = z \cdot \mathrm{T}_f \mathcal{L}_X \subseteq \mathcal{H}$;
- the set of all tangent spaces to $\mathcal{L}_X$ forms a finite-dimensional family; thus $\mathcal{L}_X$ is ruled by a finite-dimensional family of linear subspaces.

Thus we see that the geometry of $\mathcal{L}_X$ is very tightly constrained. The above theorem is actually equivalent [Giv04, Theorem 1] to the following three fundamental results in Gromov–Witten theory: the string equation, the dilaton equation and the topological recursion relations. More generally, the Lagrangian cone can be used to conveniently express statements which would be exceedingly cumbersome to phrase otherwise. For more on this, see [Giv01a], [CI14].

Finally, we note that the dilaton shift $\mathbf{q}(z) = \mathbf{t}(z) - z\mathbb{1}_X$ is an essential part of the theory; for instance, $\mathcal{L}_X$ is not even a cone in the $\mathbf{t}(z)$ co-ordinates.

4.2.3. **Fundamental solution matrix**. There is one more object in Gromov–Witten theory which we must define. The *fundamental solution matrix* is a family of symplectic operators on the Givental space $\mathcal{H}$ (so named because it encodes a fundamental set of solutions to the quantum differential equations [Dub96]). For our purposes it depends on a parameter $\mathbf{q}(z) \in \mathcal{H}_+$, and is given by:

$$S_{\mathbf{t}(z)}(f) = f + \sum_{\beta,n} \frac{Q^\beta}{n!} \left\langle \left(\frac{f}{z - \psi_0} \right), \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_n), \varphi_\gamma \right\rangle_{0,n+2,\beta}^{X} \cdot \varphi^\gamma$$

Here the insertion $f \in \mathcal{H}$ is expanded linearly in the z and φ_α, and $\mathbf{t}(z)$ is the dilaton-shifted element corresponding to $\mathbf{q}(z)$ (we write $S_{\mathbf{t}(z)}$ instead of $S_{\mathbf{q}(z)}$ to keep our notation compatible with standard usage). As with the Lagrangian cone, the fundamental solution matrix has deep connections to physics, and has been the focus of intense study. We will not attempt to say more than this here; the interested reader should consult [Pan98] and [CK99, §10].

In this chapter we will view S as a single endomorphism of the trivial $\mathcal{H}$-bundle over $\mathcal{H}_+$

$$\begin{array}{ccc} \mathcal{H}_+ \times \mathcal{H} & \xrightarrow{\quad S \quad} & \mathcal{H}_+ \times \mathcal{H} \\ & \searrow \quad \swarrow & \\ & \mathcal{H}_+ & \end{array}$$

where the endomorphism $\mathcal{H} \to \mathcal{H}$ over $\mathbf{q}(z) \in \mathcal{H}_+$ is given by $S_{\mathbf{t}(z)}$. We can also view the Lagrangian cone as a submanifold of $\mathcal{H}_+ \times \mathcal{H}$ by doubling the base co-ordinate:

$$\mathcal{L}_X = \left\{ (\mathbf{q}(z), \mathbf{q}(z), \mathbf{p}(z)) \;\middle|\; \mathbf{p}(z) = \mathrm{d}\mathcal{F}^0_X(\mathbf{q}(z)) \right\} \subseteq \mathcal{H}_+ \times \mathcal{H}$$

Thus, we can define the transform $S(\mathcal{L}_X) \subseteq \mathcal{H}_+ \times \mathcal{H}$ of $\mathcal{L}_X$ by S without having to specify a parameter $\mathbf{q}(z)$. This will be important for the statement of our main result.

4.2.4. **Relative stable maps**. The final ingredient which we need to explain is the theory of relative stable maps. Given a smooth projective variety Z and a smooth hypersurface $Y \subseteq Z$, the moduli space of relative stable maps parametrises stable maps in Z with fixed tangency orders to Y at the marked points. If there are n marked points then this tangency information is encoded in a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers. The resulting moduli space

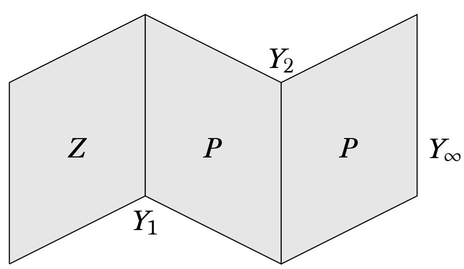
$$\overline{\mathcal{M}}_{0,\alpha}(Z|Y, \beta)$$

should parametrise stable maps to Z such that the ith marked point has tangency order α_i to the divisor Y. This data must satisfy the obvious numerical condition $\Sigma_i \alpha_i = Y \cdot \beta$. The question of how to define these spaces rigorously is a non-trivial one; the problem with the naïve approach described above is that the deformation theory can become extremely wild when there are components of the source curve mapping into Y; this wildness means that the usual construction of the virtual fundamental class no longer works, so these spaces cannot be used to define invariants.

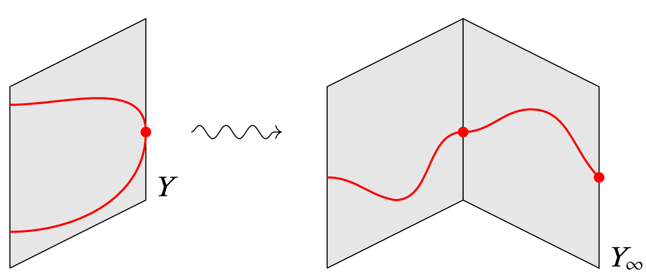
The earliest solution to this problem, due to J. Li and following ideas first developed in symplectic geometry, is to allow the target Z to degenerate into a so-called expanded degeneration $Z[l]$ [Li01, Li02]. The space $Z[l]$ is constructed from Z by gluing on a chain of l copies of the projective completion of the normal bundle to Y in Z:

$$P = \mathbb{P}_Y \left(\mathrm{N}_{Y|Z} \oplus \mathcal{O}_Y \right)$$

The picture is as follows (which illustrates the case $Z[2]$):



The idea is that, whenever a component of the source curve starts to fall into the divisor, the target "bubbles" off an extra copy of P, and the internal component is then mapped (transversely) into P.



Two such maps into P are identified if they differ by an element of the group $\mathbb{C}^*$ of automorphisms of P given by rescalings of the fibre. As illustrated above, the resulting map to $Z[l]$ is transverse in a very strong sense: the only points of the curve which map to the infinity divisor are the markings x_i, and they do so with the correct tangency order α_i. On the other hand, the curve can only map to the singular locus at a finite number of isolated nodal points, and for each node the tangency orders of the two adjacent branches of the curve to the singular locus must be equal. This transversality condition, usually called predeformability, ensures that the resulting moduli space has the correct virtual dimension. An extremely careful analysis of the deformation theory of this new space then shows that a virtual class can be defined [Li02]. Integrals against this virtual class are called *relative Gromov–Witten invariants* of (Z, Y). In our applications we will always have $Z = X \times \mathbb{P}^1$ and $Y = X \times \infty$. In this case the normal bundle of Y in Z is trivial, so $P \cong X \times \mathbb{P}^1 = Z$ and thus all the levels of the expanded degeneration, including level 0, are isomorphic.

We will assume that the reader is reasonably familiar with relative stable maps; all the facts which we will use can be found in §§2-3 of [GV05], which also serves as a good introduction to relative Gromov–Witten theory.

Remark 4.2.3. More recently, the theory of *logarithmic stable maps*, as developed by D. Abramovich, Q. Chen, M. Gross and B. Siebert, has provided an alternative (and significantly more general) approach to relative stable maps [GS13] [Che14] [AC14]. We expect that the computations we carry out here will carry over to the log setting, once a suitable localisation formula has been established for log stable maps. Indeed, log Gromov–Witten theory relative a simple normal crossings divisor seems to be the correct generality in which to apply the construction given in this chapter.

4.2.5. **Statement of the main result**. We are finally in a position to state our main result. Let X be a smooth projective variety. For $\beta \in \mathrm{H}_2^+(X)$ and $n \geq 0$, consider the moduli space

$$\overline{\mathcal{M}}_{0,n,(1)}\left((X \times \mathbb{P}^1 \mid X \times \infty), (\beta, 1)\right)$$

of relative stable maps to $(X \times \mathbb{P}^1, X \times \infty)$ of class $(\beta, 1)$, where the first n marked points $x_1, \ldots, x_n$ have tangency 0 with the divisor, and the last marked point x_∞ has tangency 1. There is a natural $\mathbb{C}^*$-action on this moduli space induced by the action on the target $X \times \mathbb{P}^1$ (acting trivially on the first factor and with weight -1 on the second). Consider the following class in the equivariant cohomology of the moduli space

$$\Theta_{\beta,n}(\mathbf{t}(z)) = (-z) \cdot \prod_{i=1}^{n} \mathrm{ev}_i^*(\mathbf{t}(\psi_i))$$

where z is the equivariant parameter. Then we have:

Proposition 4.2.4.

$$(\mathrm{ev}_\infty)_* \left(\sum_{\beta,n} \frac{Q^\beta}{n!} \cdot \Theta_{\beta,n}(\mathbf{t}(z)) \right) = S(\mathcal{L}_X)|_{\mathbf{q}(z)} \tag{4.2.1}$$

where $\mathbf{q}(z)$ is the dilaton-shifted co-ordinate corresponding to $\mathbf{t}(z)$.

The proof will be given in §4.3; for the moment let us explain the statement. We view ev_∞ as a map

$$\mathrm{ev}_\infty \colon \coprod_{\beta,n} \overline{\mathcal{M}}_{0,n,(1)} \Big((X \times \mathbb{P}^1 \mid X \times \infty), (\beta, 1) \Big) \longrightarrow X \times \infty = X$$

so that the target of the push-forward $(\mathrm{ev}_\infty)_*$ is the equivariant cohomology of X with respect to the trivial torus action. But this is just:

$$\mathrm{H}^*(X) \otimes \Lambda[z] = \mathcal{H}_+ \subseteq \mathcal{H}$$

On the other hand, $S(\mathcal{L}_X)$ naturally lives inside the total space of the trivial bundle $\mathcal{H}_+ \times \mathcal{H} \to \mathcal{H}_+$ (see the discussion at the end of §4.2.3 above); therefore when we write $S(\mathcal{L}_X)$ in equation (4.2.1), we really mean its projection along $\pi_2 \colon \mathcal{H}_+ \times \mathcal{H} \to \mathcal{H}$. Another way to say this is that for a fixed $\mathbf{q}(z) \in \mathcal{H}_+$, with dilaton-shifted co-ordinate $\mathbf{t}(z)$, the push-forward of the left-hand side of (4.2.1) is equal to $S_{\mathbf{t}(z)}(\mathcal{L}_X|_{\mathbf{q}(z)})$.

An immediate corollary of the above result is that $S(\mathcal{L}_X) \subseteq z \cdot \mathcal{H}_+$ rather than just $\mathcal{H}$. For an application of this, as well as a deeper exploration of the "hidden polynomiality" arising from our construction, see §4.4.

Remark 4.2.5. The total transform $S(\mathcal{L}_X)$ has a geometric interpretation as a family of *ancestor cones*; see [CG07, Appendix 2].

Remark 4.2.6. Notice that for any choice of β, the curve class $(\beta, 1)$ is non-zero. Hence the sum in Proposition 4.2.4 is over *all* β and n. This is in contrast to the sum which appears in the definition of the Lagrangian cone in §4.2.2, which is only over the stable range, i.e. excludes the cases $(\beta, n) = (0,0)$ and $(0,1)$. This difference will become important during the proof of Proposition 4.2.4.

We will assume that the reader is familiar with the space of relative stable maps, and in particular with the torus localisation formula, established in [GV05] whenever the divisor is fixed pointwise by the action (as is the case for us). We will write X_0 and X_∞ for $X \times 0$ and $X \times \infty$, viewing them either as divisors in $X \times \mathbb{P}^1$ or in $X[l]$, as appropriate.

4.3.1. **Identifying the fixed loci.** The proof proceeds by $\mathbb{C}^*$-localisation. The $\mathbb{C}^*$-fixed loci of the moduli space are indexed by graphs of the following form:

These correspond to splittings of the source curve into two pieces: a piece C_0 which maps to X_0 and a piece C_∞ which maps to X_∞ (and hence, in general, into the higher levels of the expanded degeneration); the two pieces are joined by a rational component which maps isomorphically onto a $\mathbb{P}^1$-fibre of $X \times \mathbb{P}^1$. The marking x_∞ always belongs to the second piece since it must map to the infinity divisor. The other choices – of degrees β_0 and β_∞ for the two pieces, and of a partition $A_0 \sqcup A_\infty = \{x_1, \ldots, x_n\}$ of the non-relative markings – are free. The fixed locus corresponding to this data is isomorphic to

$$\overline{\mathcal{M}}_{0,A_0\cup\{q_0\}}(X,\beta_0) \times_X \overline{\mathcal{M}}_{0,A_\infty,(1),(1)}\left(X \times \mathbb{P}^1 \mid (X_0 + X_\infty), \beta_\infty\right)_\sim$$

with virtual fundamental class induced by the virtual classes of the two factors; this is part of the statement of the virtual localisation theorem in [GV05]. Here the second factor is a moduli space of stable maps to the non-rigid target; see [GV05, §2.4]. The notation is supposed to indicate that there is a set A_∞ of non-relative markings, a single marking q_∞ which maps to X_0 with tangency 1, and a single marking x_∞ which maps to X_∞ with tangency 1. The fibre product is taken with respect to the evaluations at q_0 and q_∞ on each side. The Euler class of the virtual normal bundle is equal [GV05, Theorem 3.6 and Example 3.7] to

$$(-z)(-z - \psi_{q_0})(z - \psi_{q_\infty})$$

which obviously splits into a product of classes supported on the two factors. We should briefly explain these: $-z$ arises from the deformations of the map on the rational bridge, $-z - \psi_{q_0}$ arises from the smoothing of the node connecting the rational bridge to C_0 and $z - \psi_{q_\infty}$ is a target psi class, which arises from the smoothing of the target singularity connecting the level 0 piece and the level 1 piece of the expanded degeneration. Here we have used the identification of the target psi class with a multiple of the psi class on one of the relative markings [Gat03a, Construction 5.1.17]. There is also a contribution arising from the smoothing of the node connecting the rational bridge to C_∞, but this is canceled out by the local obstruction at that node (see [GV05, §3.8]).

Note that for certain choices of $(\beta_0, A_0 \mid \beta_\infty, A_\infty)$ the moduli spaces which we have written down above do not exist, because the data defining them is not stable. In these degenerate cases, we still have fixed loci; it is simply that one (or both) of the factors becomes trivial. Hence we must deal with these separately. The possible situations are enumerated below.

Case 1: $(\beta, n) = (0, 0)$. This is the maximally degenerate case. The fixed locus is just X, which has virtual codimension 0; there is no virtual normal bundle.

Case 2: $(\beta, n) = (0, 1)$ *and* $n_\infty = 0$. In this case the fixed locus is again just X, with a single marked point x_1 mapped to X_0 and another marked point x_∞ mapped to X_∞ (there is no expansion of the target). The virtual codimension is 1, and the Euler class of the virtual normal bundle is $-z$.

Case 3: $n \geq 1$ *and* $(\beta_0, n_0) = (0, 0)$. In this case the fixed locus is a moduli space of relative maps to the non-rigid target, with $n+2$ marked points. The virtual codimension is 1, and the virtual normal bundle contribution is $z - \psi_{q_\infty}$.

Case 4: $n \geq 1$ *and* $(\beta_0, n_0) = (0, 1)$. Here the fixed locus is the same as the one in the previous case, but it now has virtual codimension 2 because there is a marked point at the X_0 end of the rational bridge; the Euler class of the virtual normal bundle is $-z(z - \psi_{q_\infty})$.

Case 5: $n \geq 2$ *and* $(\beta_\infty, n_\infty) = (0, 0)$. In this case the fixed locus is just the moduli space of stable maps to X with $n+1$ markings. The virtual codimension is 2, and the Euler class of the virtual normal bundle is $-z(-z - \psi_{q_0})$.

4.3.2. **Comparison lemma for psi classes**. We now need to calculate the contributions to the push-forward from each of these fixed loci. A priori this is difficult, because the fixed loci involve moduli spaces of relative stable maps to the non-rigid target, which are in general hard to understand. However, in genus zero, a beautiful result of A. Gathmann says that these moduli spaces are in fact virtually birational to the underlying moduli spaces of stable maps to X. To be more precise: there is a projection map

$$\pi\colon \overline{\mathcal{M}}_{0,n_\infty,(1),(1)}\left(X\times\mathbb{P}^1|(X_0+X_\infty),\beta_\infty\right)_\sim \to \overline{\mathcal{M}}_{0,n_\infty+2}(X,\beta_\infty)$$

induced by the collapsing map from the non-rigid target to X, and [Gat03a, Theorem 5.2.7] shows that this map respects the virtual classes:

$$\pi_*[\overline{\mathcal{M}}_{0,n_\infty,(1),(1)}\left(X\times\mathbb{P}^1|(X_0+X_\infty),\beta_\infty\right)_\sim]^{\mathrm{virt}} = [\overline{\mathcal{M}}_{0,n_\infty+2}(X,\beta_\infty)]^{\mathrm{virt}}$$

This result goes a long way towards making these invariants computable. However there is still a problem: the map π may contract many components of the source curve, and hence does not in general preserve the psi classes. Consequently, descendant invariants (which certainly appear in our discussion) are still complicated to compute, because one has to keep track of how psi classes pull back. It turns out, however, that $X\times\mathbb{P}^1$ is special in this respect.

Lemma 4.3.1. The map π cannot contract any component of the source curve which contains a marking.

Proof. The components contracted by π are those with two or fewer special points which are mapped into a fibre of $P = X\times\mathbb{P}^1$ over X. Let C' be such a component. Since it has two or fewer special points, the map f must be non-constant on C' (by stability), and hence there is at least one point of C' which maps to X_∞ and at least one point which maps to X_0. Thus, C' contains exactly two special points, which must map to the special divisors of the non-rigid target.

Now suppose for a contradiction that some marking x_i belongs to C'. If x_i is a non-relative marking then we immediately arrive at a contradiction, since such a marking cannot map into any special divisor. Otherwise, $x_i = q_\infty$ or x_∞ and so is mapped into X_0 or X_∞, respectively; without loss of generality we may suppose $x_i = q_\infty$. By the stability condition for relative maps, there must exist some other component of the source curve which maps with positive degree into the same level of the non-rigid target as C'. But this would necessarily touch X_0, which is a contradiction since q_∞ is the only point of the source curve which is allowed to map to X_0 (here we are using something special about the geometry of $X\times\mathbb{P}^1$; for non-trivial $\mathbb{P}^1$-bundles over X, it is no longer true that a component of the source curve which touches X_∞ must also touch X_0). □

Corollary 4.3.2. $\pi^*\psi_i = \psi_i$ for any $i \in \{1, \ldots, n_\infty + 2\}$. Thus, we can identify any non-rigid invariant of $(X \times \mathbb{P}^1, X_0 + X_\infty)$ with the corresponding invariant of X.

4.3.3. **Calculating the contributions.** We are now in a position to calculate the contributions to the push-forward. We fix (β, n) and look at the fixed loci of the corresponding moduli space. Ignoring the degenerate cases for the moment, we must sum over *stable splittings* $(\beta_0, A_0 \mid \beta_\infty, A_\infty)$ of (β, n). We may phrase this as summing over splittings (β_0, β_∞) of β and (n_0, n_∞) of n, with a factor of $\binom{n}{n_0} = \binom{n}{n_\infty}$ introduced to account for the choice of which marked points to put in A_0 and which to put in A_∞. Thus the contribution

$$\frac{Q^\beta}{n!}(\mathrm{ev}_\infty)_*\left((-z)\cdot\prod_{i=1}^{n}\mathrm{ev}_i^*(\mathbf{t}(\psi_i))\right)$$

is equal to:

$$\begin{aligned}
&\frac{Q^\beta}{n!}\sum_{\substack{\beta_0+\beta_\infty=\beta\\ n_0+n_\infty=n}}\binom{n}{n_\infty}\langle \mathbf{t}(\psi_1),\ldots,\mathbf{t}(\psi_{n_0}),\left(\frac{\varphi_\alpha}{-z-\psi_{q_0}}\right)\rangle^X_{0,n_0+1,\beta_0}\cdot\\
&\qquad\qquad\langle\left(\frac{\varphi^\alpha}{z-\psi_{q_\infty}}\right),\mathbf{t}(\psi_1),\ldots,\mathbf{t}(\psi_{n_\infty}),\varphi_\gamma\rangle^X_{0,n_\infty+2,\beta_\infty}\cdot\varphi^\gamma\\
=&\sum_{\substack{\beta_0+\beta_\infty=\beta\\ n_0+n_\infty=n}}\left(\frac{Q^{\beta_0}}{n_0!}\langle\mathbf{t}(\psi_1),\ldots,\mathbf{t}(\psi_{n_0}),\left(\frac{\varphi_\alpha}{-z-\psi_{q_0}}\right)\rangle^X_{0,n_0+1,\beta_0}\right)\cdot\\
&\qquad\qquad\left(\frac{Q^{\beta_\infty}}{n_\infty!}\langle\left(\frac{\varphi^\alpha}{z-\psi_{q_\infty}}\right),\mathbf{t}(\psi_1),\ldots,\mathbf{t}(\psi_{n_\infty}),\varphi_\gamma\rangle^X_{0,n_\infty+2,\beta_\infty}\cdot\varphi^\gamma\right)
\end{aligned}$$

There are also the contributions from the degenerate fixed loci, enumerated in §4.3.1 above. We now calculate these.

Case 1: $(\beta, n) = (0, 0)$. This gives a single contribution, which is

$$-z(\mathrm{ev}_\infty)_*(\mathbb{1}_X) = -z\mathbb{1}_X$$

Case 2: $(\beta, n) = (0, 1)$ *and* $n_\infty = 0$. This also gives a single contribution, which is

$$(\mathrm{ev}_\infty)_*(\mathrm{ev}_1^*\,\mathbf{t}(\psi_1)) = \mathbf{t}(z)$$

here we have used the fact that the psi class ψ_1 restricts to a trivial class on the fixed locus with non-trivial weight z, so the equivariant class ψ_1 gets identified with z.

Case 3: $n \geq 1$ and $(\beta_0, n_0) = (0,0)$. Here we get a contribution for each (β, n) with $n \geq 1$. The contribution is:

$$\frac{Q^{\beta_\infty}}{n_\infty!}\langle \left(\frac{-z\mathbb{1}_X}{z-\psi_{q_\infty}}\right), \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_{n_\infty}), \varphi_\gamma\rangle^X_{0,n_\infty+2,\beta} \cdot \varphi^\gamma$$

Case 4: $n \geq 1$ and $(\beta_0, n_0) = (0,1)$. We get a contribution for each (β, n) with $n \geq 1$, and the contribution is

$$\frac{Q^{\beta_\infty}}{n_\infty!}\langle \left(\frac{\mathbf{t}(z)}{z-\psi_{q_\infty}}\right), \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_{n_\infty}), \varphi_\gamma\rangle^X_{0,n_\infty+2,\beta_\infty} \cdot \varphi^\gamma$$

where again we have used the fact that the class ψ_0 restricts to the pure weight class z on the fixed locus.

Case 5: $n \geq 2$ and $(\beta_\infty, n_\infty) = (0,0)$. Here we get a contribution for each (β, n) with $n \geq 2$, and the contribution is:

$$\frac{Q^{\beta_0}}{n_0!}\langle \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_{n_0}), \left(\frac{\varphi_\gamma}{-z-\psi_{q_0}}\right)\rangle^X_{0,n_0+1,\beta_0} \cdot \varphi^\gamma$$

4.3.4. **Putting everything together**. If we sum together all the terms computed in the previous section, we obtain:

$$\begin{aligned}
(\mathbf{t}(z) - z\mathbb{1}_X) &+ \sum_{\beta_0,n_0} \frac{Q^{\beta_0}}{n_0!}\langle \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_{n_0}), \left(\frac{\varphi_\alpha}{-z-\psi_{q_0}}\right)\rangle^X_{0,n_0+1,\beta_0} \cdot \varphi^\alpha \\
&+ \left(\sum_{\beta_0,n_0} \frac{Q^{\beta_0}}{n_0!}\langle \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_{n_0}), \left(\frac{\varphi_\alpha}{-z-\psi_{q_0}}\right)\rangle^X_{0,n_0+1,\beta_0}\right) \cdot \\
&\left(\sum_{\beta_\infty,n_\infty} \frac{Q^{\beta_\infty}}{n_\infty!}\langle \left(\frac{\varphi^\alpha}{z-\psi_{q_\infty}}\right), \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_{n_\infty}), \varphi_\gamma\rangle^X_{0,n_\infty+2,\beta_\infty} \cdot \varphi^\gamma\right) \\
&+ \sum_{\beta_\infty,n_\infty} \frac{Q^{\beta_\infty}}{n_\infty!}\langle \left(\frac{\mathbf{t}(z) - z\mathbb{1}_X}{z-\psi_{q_\infty}}\right), \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_n), \varphi_\gamma\rangle^X_{0,n_\infty+2,\beta_\infty} \cdot \varphi^\gamma
\end{aligned}$$

Using $\mathbf{q}(z) = \mathbf{t}(z) - z\mathbb{1}_X$ and grouping the final two terms together, we see that this is equal to:

$$\begin{aligned}
\mathbf{q}(z) + &\sum_{\beta_0,n_0} \frac{Q^{\beta_0}}{n_0!}\langle \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_{n_0}), \left(\frac{\varphi_\alpha}{-z-\psi_{q_0}}\right)\rangle^X_{0,n_0+1,\beta_0} \cdot \varphi^\alpha \\
&+ \sum_{\beta_\infty,n_\infty} \frac{Q^{\beta_\infty}}{n_\infty!}\left\langle \frac{1}{z-\psi_{q_\infty}} \cdot \left(\mathbf{q}(z) + \sum_{\beta_0,n_0} \frac{Q^{\beta_0}}{n_0!}\langle \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_{n_0}), \left(\frac{\varphi_\alpha}{-z-\psi_{q_0}}\right)\rangle^X_{0,n_0+1,\beta_0} \cdot \varphi^\alpha\right),\right. \\
&\qquad\qquad\qquad \left. \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_{n_\infty}), \varphi_\gamma \right\rangle^X_{0,n_\infty+2,\beta_\infty} \cdot \varphi^\gamma
\end{aligned}$$

But this is equal to

$$\mathcal{L}_X|_{\mathbf{q}(z)} + \sum_{\beta_\infty, n_\infty} \frac{Q^{\beta_\infty}}{n_\infty!} \langle \left(\frac{\mathcal{L}_X|_{\mathbf{q}(z)}}{z - \psi_{q_\infty}} \right), \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_{n_\infty}), \varphi_\gamma \rangle^X_{0, n_\infty+2, \beta_\infty} \cdot \varphi^\gamma = S(\mathcal{L}_X)|_{\mathbf{q}(z)}$$

as claimed. This completes the proof of Proposition 4.2.4.

Remark 4.3.3. It is perhaps worth comparing our computation to the computation carried out in [Coa08]. There, the moduli space under consideration is the space of ordinary stable maps to $X \times \mathbb{P}^1$; Coates restricts to an open substack of this space, consisting of stable maps such that only a single point of the curve is mapped to X_∞. He then applies torus localisation and pushes forward from the (proper) fixed loci. From our point of view, the loci from which he pushes forward are the degenerate loci which appear as Case 5 in §4.3.1 above. The special cases which he calls Case 2 and Case 3 are what we call Case 2 and Case 1, respectively. Our non-special case, which contributes a product of invariants from stable maps to X and stable maps to the non-rigid target, does not appear in his setting; nor do our special cases 3 and 4.

4.4. Variants and applications

Since an equivariant push-forward must take values in $\mathrm{H}^*(X) \otimes \Lambda[z] = \mathcal{H}_+$, an immediate consequence of Proposition 4.2.4 is the following:

Theorem 4.4.1. $S(\mathcal{L}_X) \subseteq z \cdot \mathcal{H}_+$.

This is somewhat surprising, since a priori we only know that $S(\mathcal{L}_X) \subseteq \mathcal{H}$, and indeed both S and $\mathcal{L}_X$ involve many non-positive powers of z. What Theorem 4.4.1 says is that the coefficients of these non-positive powers cancel out when we take $S(\mathcal{L}_X)$; this translates into a sequence of universal relations for the Gromov–Witten invariants. Calculating the coefficients of z^{-k} explicitly, we obtain for $k \geq 2$ and $\mathbf{q}(z) \in \mathcal{H}_+$

$$\Bigg(\langle\langle \psi_1^{k-1} \mathbf{q}(\psi_1), \varphi_\alpha \rangle\rangle^X_{0,2} + (-1)^k \langle\langle \varphi_\alpha \psi_1^{k-1} \rangle\rangle^X_{0,1} + \sum_{r=0}^{k-2} (-1)^{1+r} \langle\langle \varphi_\gamma \psi_1^r \rangle\rangle^X_{0,1} \cdot \langle\langle \varphi^\gamma \psi_1^{k-2-r}, \varphi_\alpha \rangle\rangle^X_{0,2} \Bigg) (\mathbf{t}(\psi)) \cdot \varphi^\alpha = 0$$

where we have used the correlator notation:

$$\langle\langle \varphi_{\alpha_1} \psi_1^{k_1}, \ldots, \varphi_{\alpha_r} \psi_r^{k_r} \rangle\rangle^X_{0,r}(\mathbf{t}(\psi)) := \sum_{\beta, n} \frac{Q^\beta}{n!} \langle \varphi_{\alpha_1} \psi_1^{k_1}, \ldots, \varphi_{\alpha_r} \psi_r^{k_r}, \mathbf{t}(\psi_{r+1}), \ldots, \mathbf{t}(\psi_{r+n}) \rangle^X_{0, n+r, \beta}$$

These equations appear to be equivalent to the reconstruction relation [LP04, Equation (2)], combined with the dilaton equation.

Remark 4.4.2. Theorem 4.4.1 can be viewed as a generalisation of one of the fundamental results in the quantisation formalism, namely that the J-function is inverse to the fundamental solution matrix; see Remark 4.4.4 below.

In this section we will now extend the above line of argument, exploiting the "hidden polynomiality" implicit in our construction. We obtain new proofs and generalisations of several foundational results concerning both the fundamental solution matrix and the Lagrangian cone.

4.4.1. **The fundamental solution matrix and its adjoint**. Looking at the definition given in §4.2.3, we see that we can regard $S_{\mathbf{t}(z)}$ as a power series in z^{-1} with coefficients in $\mathrm{End}(\mathrm{H}^*(X))$:

$$S_{\mathbf{t}(z)} \in \mathrm{End}(\mathrm{H}^*(X))[\![z^{-1}]\!]$$

We will write $S_{\mathbf{t}(z)}(z)$ to emphasise this point of view. The adjoint ${S_{\mathbf{t}(z)}}^*(z)$ is defined by taking the adjoints, term-by-term, of the coefficients of $S_{\mathbf{t}(z)}(z)$ (with respect to the Poincaré pairing on $\mathrm{H}^*(X)$). It is easy to check that, for $v \in \mathrm{H}^*(X)$:

$$(4.4.1) \qquad {S_{\mathbf{t}(z)}}^*(z)(v) = v + \sum_{\beta,n} \frac{Q^\beta}{n!} \langle v, \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_n), \left(\frac{\varphi_\alpha}{z-\psi}\right) \rangle^X_{0,n+2,\beta} \cdot \varphi^\alpha$$

An important feature of the theory [Giv96] is that when $\mathbf{t}(z) = \tau$, the operators $S_\tau(z)$ and ${S_\tau}^*(-z)$ are inverse to each other; this is in fact equivalent to the statement that $S_\tau(z)$ is a symplectomorphism [CPS13, §3.1]. We now generalise this fact to arbitrary $\mathbf{t}(z)$, based on a slight modification of the construction used in Proposition 4.2.4.

Proposition 4.4.3. ${S_{\mathbf{t}(z)}}^*(-z) = S_{\mathbf{t}(z)}(z)^{-1}$.

Proof. We first note that it is sufficient to prove:

$$(4.4.2) \qquad S_{\mathbf{t}(z)}(z) \circ {S_{\mathbf{t}(z)}}^*(-z) = \mathrm{Id}_{\mathrm{H}^*(X)}$$

Indeed, the operators $S_{\mathbf{t}(z)}(z)$ and ${S_{\mathbf{t}(z)}}^*(-z)$ can be viewed as finite-dimensional matrices over the field of Laurent series $\Lambda((z^{-1}))$. If (4.4.2) holds then both these matrices have maximal rank, and therefore we also have:

$${S_{\mathbf{t}(z)}}^*(-z) \circ S_{\mathbf{t}(z)}(z) = \mathrm{Id}_{\mathrm{H}^*(X)}$$

Thus it remains to show (4.4.2). We consider the following moduli space

$$\overline{\mathcal{M}}_{0,n,(1),(1)}\left((X \times \mathbb{P}^1 \mid X_0 + X_\infty), (\beta, 1)\right)$$

which has a single marked point x_0 mapping to X_0, a single marked point x_∞ mapping to X_∞, and a collection of other markings $x_1, \ldots, x_n$ which carry no tangency conditions.

Since the divisor is now disconnected, we must be slightly careful about what we mean by the space above. For our purposes, the allowed automorphisms act separately on the fibres of the expanded degeneration over X_0 and X_∞. The stability condition is also imposed separately. As such, each expansion is now indexed by two integers, l_0 and l_∞, giving the lengths of the expansion over X_0 and X_∞ respectively. This is close to the approach taken in [FP05]. One can view this moduli space as the fibre product:

$$\overline{\mathcal{M}}_{0,n+1,(1)}\left(X \times \mathbb{P}^1 | X_0, (\beta, 1)\right) \times_{\overline{\mathcal{M}}_{0,n+2}(X \times \mathbb{P}^1, (\beta,1))} \overline{\mathcal{M}}_{0,n+1,(1)}\left(X \times \mathbb{P}^1 | X_\infty, (\beta, 1)\right)$$

Taking the definition this way ensures that, when we localise, the fixed loci are fibre products of moduli spaces of relative stable maps to the non-rigid target. Furthermore since the stability condition is imposed separately over X_0 and X_∞, the proof of Lemma 4.3.1 still applies. An analogous computation to the one given in §4.3 then shows that, for $v \in \mathrm{H}^*(X)$:

$$(\mathrm{ev}_\infty)_* \left(\sum_{\beta,n} \frac{Q^\beta}{n!} \cdot \mathrm{ev}_0^*(v) \cdot \prod_{i=1}^{n} \mathrm{ev}_i^* \, \mathbf{t}(\psi_i)\right) = S_{\mathbf{t}(z)}(z) \left({S_{\mathbf{t}(z)}}^*(-z)(v)\right)$$

Since this is an equivariant push-forward, we see that $S_{\mathbf{t}(z)}(z) \circ {S_{\mathbf{t}(z)}}^*(-z)$ is a polynomial in z with coefficients in $\mathrm{End}(\mathrm{H}^*(X))$. On the other hand it is obvious from the definitions that it is also a power series in z^{-1}. Thus $S_{\mathbf{t}(z)}(z) \circ {S_{\mathbf{t}(z)}}^*(-z)$ is constant in z, and since the constant term is clearly the identity this completes the proof. □

Remark 4.4.4. As noted previously, Proposition 4.4.3 is a generalisation of the following fundamental fact for $\tau \in \mathrm{H}^*(X)$:

$${S_\tau}^*(-z) = S_\tau(z)^{-1}$$

I would like to thank M. Shoemaker for pointing out that one can also view Theorem 4.4.1 as a generalisation of this result. Indeed, when $\mathbf{t}(z) = \tau$ we can use the string equation to show that

$$\mathcal{L}_X|_{\mathbf{q}(z)} = {S_\tau}^*(-z)(-z) \tag{4.4.3}$$

where $\mathbf{q}(z) = \tau - z$. Thus we find:

$$S(\mathcal{L}_X)|_{\mathbf{q}(z)} = S_\tau(\mathcal{L}_X|_{\mathbf{q}(z)}) = S_\tau(z) \circ {S_\tau}^*(-z)(-z) = -z \in z \cdot \mathcal{H}_+$$

Our result can be viewed as a generalisation of this to arbitrary $\mathbf{t}(z)$. The original proof does not apply in this more general setting, because it relies on an application of the string equation which produces additional unwanted terms when $\mathbf{t}(z)$ involves higher powers of z. In particular, the identification (4.4.3) no longer holds, which explains why we end up with two different generalisations.

4.4.2. **Properties of the Lagrangian cone**. Here we reprove two fundamental facts concerning the Lagrangian cone. First, we modify the previous construction to give a concrete proof that $\mathcal{L}_X$ is Lagrangian (though it should be noted that this also follows from the general fact that the graph of any closed 1-form is Lagrangian).

Proposition 4.4.5. $\mathcal{L}_X$ is Lagrangian.

Proof. Let $\mathbf{q}(z) \in \mathcal{H}_+$ be a point in the base and let $f = \mathcal{L}_X|_{\mathbf{q}(z)} \in \mathcal{H}$ be the point on the cone lying over $\mathbf{q}(z)$. We must show that $\mathrm{T}_f\,\mathcal{L}_X$ is a Lagrangian subspace of $\mathcal{H}$. First let us describe the points of $\mathrm{T}_f\,\mathcal{L}_X$. Recall that f is given by:

$$f = \mathcal{L}_X|_{\mathbf{q}(z)} = \mathbf{q}(z) + \sum_{\beta,n} \frac{Q^\beta}{n!} \langle \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_n), \left(\frac{\varphi_\gamma}{-z-\psi}\right) \rangle^X_{0,n+1,\beta} \cdot \varphi^\gamma$$

Since $\mathcal{L}_X$ is the graph of the section $\mathrm{d}\mathcal{F}^0_X$, the tangent space $\mathrm{T}_f\,\mathcal{L}_X$ is spanned by the partial derivatives of the above expression in the $\mathcal{H}_+$-co-ordinates. Given such a co-ordinate q^α_k the corresponding derivative is:

$$\varphi_\alpha z^k + \sum_{\beta,n} \frac{Q^\beta}{n!} \langle \varphi_\alpha \psi^k, \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_n), \left(\frac{\varphi_\gamma}{-z-\psi}\right) \rangle^X_{0,n+2,\beta} \cdot \varphi^\gamma$$

Thus the tangent space consists of vectors in $\mathcal{H}$ of the form

$$\mathbf{r}(z) + \sum_{\beta,n} \frac{Q^\beta}{n!} \langle \mathbf{r}(\psi), \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_n), \left(\frac{\varphi_\gamma}{-z-\psi}\right) \rangle^X_{0,n+2,\beta} \cdot \varphi^\gamma$$

for $\mathbf{r}(z) \in \mathcal{H}_+$. On the other hand, if we look at the expression (4.4.1) given earlier for ${S_{\mathbf{t}(z)}}^*(z) \in \mathrm{End}(\mathrm{H}^*(X))[\![z^{-1}]\!]$, we see that this can be extended in a natural way to give a map $\mathcal{H}_+ \to \mathcal{H}$ via

$${S_{\mathbf{t}(z)}}^*(z)(\mathbf{r}(z)) = \mathbf{r}(z) + \sum_{\beta,n} \frac{Q^\beta}{n!} \langle \mathbf{r}(\psi), \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_n), \left(\frac{\varphi_\gamma}{z-\psi}\right) \rangle^X_{0,n+2,\beta} \cdot \varphi^\gamma$$

(note that this is *different* from the extension of $S_{\mathbf{t}(z)}(z)$ to an endomorphism of $\mathcal{H}$ which we gave in §4.2.3, where we treated the insertion $\mathbf{r}(z)$ formally). Under the above definition, we see that:

$$\mathrm{T}_f\,\mathcal{L}_X = {S_{\mathbf{t}(z)}}^*(-z)(\mathcal{H}_+)$$

Fixing $\mathbf{r}(z), \mathbf{u}(z) \in \mathcal{H}_+$, we thus need to show that:

$$\Omega\Big({S_{\mathbf{t}(z)}}^*(z)(\mathbf{r}(-z)), {S_{\mathbf{t}(z)}}^*(-z)(\mathbf{u}(z))\Big) = 0$$

$$\Leftrightarrow \operatorname{Res}_{z=0} \left({S_{\mathbf{t}(z)}}^*(z)\mathbf{r}(-z), {S_{\mathbf{t}(z)}}^*(-z)(\mathbf{u}(z))\right) \mathrm{d}z = 0$$

We take the moduli space

$$\overline{\mathcal{M}}_{0,n,(1),(1)}\left((X \times \mathbb{P}^1 \mid X_0 + X_\infty), (\beta, 1)\right)$$

as before and consider the equivariant integral (against the virtual class) of the following class:

$$\sum_{\beta,n} \frac{Q^\beta}{n!} \left(\mathrm{ev}_0^*(\mathbf{r}(\psi_0)) \cdot \prod_{i=1}^{n} \mathrm{ev}_i^*(\mathbf{t}(\psi_i)) \cdot \mathrm{ev}_\infty^*(\mathbf{u}(\psi_\infty))\right)$$

Then an analogous computation to the one given in §4.3 shows cdthat this integral is equal to:

$$\left({S_{\mathbf{t}(z)}}^*(z)(\mathbf{r}(-z)), {S_{\mathbf{t}(z)}}^*(-z)(\mathbf{u}(z))\right)$$

Thus the above pairing is a polynomial in z, and so in particular the coefficient of z^{-1} vanishes. But this is precisely the residue that we needed to calculate, and the claim follows. □

Another fundamental fact about $\mathcal{L}_X$, already discussed in §4.2.2, is that:

$$(\mathrm{T}_f\, \mathcal{L}_X) \cap \mathcal{L}_X = z \cdot \mathrm{T}_f\, \mathcal{L}_X$$

To finish, we will give a direct proof of one important consequence of this fact.

Proposition 4.4.6. $f \in z \cdot \mathrm{T}_f\, \mathcal{L}_X$.

Proof. As noted before, an immediate consequence of Proposition 4.2.4 is that:

$$S_{\mathbf{t}(z)}(z)(f) \in z \cdot \mathcal{H}_+$$

Applying ${S_{\mathbf{t}(z)}}^*(-z)$ to both sides, we find that

$$f \in {S_{\mathbf{t}(z)}}^*(-z)\,(z \cdot \mathcal{H}_+)$$

where, unlike in the proof of Proposition 4.4.5, the extension of ${S_{\mathbf{t}(z)}}^*(-z)$ from $\mathrm{H}^*(X)$ to $\mathcal{H}_+ = \mathrm{H}^*(X)[z]$ is obtained by expanding linearly in z. A deep fact from the theory now says that, under this definition:

$${S_{\mathbf{t}(z)}}^*(-z)(\mathcal{H}_+) = \mathrm{T}_f\, \mathcal{L}_X$$

Some care is required here: we also saw this statement in the proof of the previous proposition, but that was for a different extension of ${S_{\mathbf{t}(z)}}^*(-z)$ which was not linear in z. Under the new extension used here, which is linear in z, the statement still holds, though it is much less trivial. Using this, we obtain:

$$f \in z \cdot {S_{\mathbf{t}(z)}}^*(-z)(\mathcal{H}_+) = z \cdot \mathrm{T}_f\, \mathcal{L}_X$$

as required. □

Remark 4.4.7. The idea of using torus localisation to prove that certain generating functions are polynomials is not new. It was used by Givental in the proof of the Mirror Theorem [Giv96] and by I. Ciocan-Fontanine and B. Kim in the proof of the wall-crossing formula for quasimap invariants [CFK16]. The disussion above constitutes a small continuation of this story.

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