6 Completeness

6.1 Cauchy sequences

Definition 6.1. A sequence (x_n) of elements of a metric space (X, ϱ) is called a *Cauchy* sequence if, given any $\varepsilon > 0$, there exists N_ε such that $\varrho(x_n, x_m) < \varepsilon$ for all $n, m > N_\varepsilon$.

Lemma 6.2. *Every convergent sequence is a Cauchy sequence.*

Proof. If $x_n \to \alpha$ then for any $\varepsilon > 0$ there exists N_ε such that $\varrho(x_n, \alpha) < \varepsilon/2$ for all $n \geq N_\varepsilon$. Applying the triangle inequality we obtain

$$\varrho(x_n, x_m) \leq \varrho(x_n, \alpha) + \varrho(\alpha, x_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all $n, m \geq N_\varepsilon$. This implies that (x_n) is a Cauchy sequence. □

6.2 Complete metric spaces

Definition 6.3. A metric space (X, ϱ) is said to be *complete* if every Cauchy sequence (x_n) in (X, ϱ) converges to a limit $\alpha \in X$.

There are incomplete metric spaces. If a metric space (X, ϱ) is not complete then it has Cauchy sequences that do not converge. This means, in a sense, that there are gaps (or missing elements) in X.

Example 6.4. Consider the rational numbers $\mathbb{Q}$ with the usual metric $\varrho(x, y) = |x - y|$. Consider a sequence of rationals x_n converging to $\sqrt{2}$ in $(\mathbb{R}, \varrho)$; specifically, we may assume $|x_n - \sqrt{2}| \leq 1/n$. (This is possible since there are rational numbers arbitrarily close to $\sqrt{2}$. Indeed, we can even generate a convergent sequence algorithmically using a simple recurrence, e.g., $x_1 = 1$, and $x_n = x_{n-1}/2 + 1/x_{n-1}$ for $n > 1$. Lemma 6.2 assures us that x_n is a Cauchy sequence. But we know from C&C that $\sqrt{2}$ is not itself a rational number. So the metric space $(\mathbb{Q}, \varrho)$ is not complete.

Every incomplete metric space can be made complete by adding new elements, which can be thought of as the missing limits of non-convergent Cauchy sequences. More precisely, we have the following theorem.

Theorem 6.5. *Let (X, ϱ) be an arbitrary metric space. Then there exists a complete metric space $(\widetilde{X}, \tilde{\varrho})$ such that*

1. *$X \subseteq \widetilde{X}$ and $\tilde{\varrho}(x, y) = \varrho(x, y)$ whenever $x, y \in X$;*

2. *for every* $\tilde{x} \in \widetilde{X}$ *there exists a sequence of elements* $x_n \in X$ *such that* $x_n \to \tilde{x}$ *as* $n \to \infty$ *in the space* $(\widetilde{X}, \tilde{\varrho})$.

Proof. The proof of this theorem is beyond the scope of the course. $\square$

The metric space $(\widetilde{X}, \tilde{\varrho})$ is said to be the *completion* of (X, ϱ). If (X, ϱ) is already complete then necessarily $X = \widetilde{X}$ and $\varrho = \tilde{\varrho}$. Note that part (1) of the theorem is saying that the original metric space (X, ϱ) is a subspace of its completion $(\widetilde{X}, \tilde{\varrho})$; Part (2) can be viewed as a minimality condition. (E.g., the complex numbers $\mathbb{C}$ with the natural metric is a complete metric space which includes $\mathbb{R}$. However, it is not a completion of $\mathbb{R}$ by part (2).) It can be shown that the completion of a metric space is unique (up to isomorphism).

Example 6.6. Let X be the set of rational numbers with the standard metric $\varrho(x, y) = |x - y|$. As we noted in the previous example, this space is not complete. The completion of $(\mathbb{Q}, \varrho)$ is the set of all real numbers $\mathbb{R}$ with the metric $\varrho(x, y) = |x - y|$. Any irrational number can be written as an infinite decimal fraction $0.a_1a_2\ldots$ or, in other words, can be identified with the Cauchy sequence 0, $0.a_1$, $0.a_1a_2$, $\ldots$ of rational numbers.

The space of real numbers $\mathbb{R}$ can be *defined* as the completion of the space of rational numbers and is then, by definition, complete.

Lemma 6.7. *Suppose* (X, ϱ) *is a complete metric space, and that* $A \subseteq X$ *is a closed subset of* X. *Then* (A, ϱ) *is a complete metric space.*

Proof. Suppose (x_n) is a Cauchy sequence in (A, ϱ). Since (x_n) is also a Cauchy sequence in (X, ϱ) it must converge to a limit $\alpha \in X$. We just need to show that $\alpha \in A$. If $\alpha = x_n$ for some n then we are done. Otherwise (since $x_n \to \alpha$) for every $\varepsilon > 0$ there exists n such that $\alpha \neq x_n \in B_\varepsilon(\alpha)$. In other words, α is a limit point of A. But A, as a closed set, contains all its limit points. $\square$

Example 6.8. The metric space $([0, 1], \varrho)$, where $\varrho(x, y) = |x - y|$ is the usual metric on reals, is a complete metric space.

Recall that, for any set S, $B(S)$ is the space of bounded functions $S \to \mathbb{R}$ equipped with the sup metric.

Theorem 6.9. $B(S)$ *is complete.*

Proof. Let $f_1, f_2, \ldots$ be a Cauchy sequence in $B(S)$. Then for any $\varepsilon > 0$ there exists N_ε such that

$$\sup_{x\in S} |f_n(x) - f_m(x)| < \varepsilon/2, \quad \text{for all } n, m \geq N_\varepsilon.$$

This implies that for each fixed $x \in S$ the numbers $f_n(x)$ form a Cauchy sequence of real numbers. Since the space of real (or complex) numbers is complete, this sequence has a limit. Let us denote this limit by $f(x)$. Then $f_n(x) \to f(x)$ for each fixed $x \in S$, that is, for any $\varepsilon > 0$ there exists an integer $M_{\varepsilon,x}$ (which may depend on x) such that

$$|f_m(x) - f(x)| < \varepsilon/2, \quad \text{for all } m \geq M_{\varepsilon,x}.$$

So far, we know that f_n converges pointwise to f. We want to show that convergence is actually uniform. (The issue here is that $M_{\varepsilon,x}$ depends on x as well as ε.) For any $x \in S$ and any $n, m \geq 1$, the triangle inequality tells us that

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)|.$$

If $n, m > N_\varepsilon$ and $m > M_{\varepsilon,x}$ then the right hand side is bounded above by ε. Therefore the left hand side is bounded by ε for all $x \in S$; indeed, given x, we can always choose m in the right hand side to be greater than both N_ε and $M_{\varepsilon,x}$. This implies that $\sup_{x\in S} |f_n(x) - f(x)| < \varepsilon$ for all $n \geq N_\varepsilon$, which means that $f_n \to f$ uniformly.

It remains to prove that f is bounded. Choosing $\varepsilon = 1$ and $n = N_\varepsilon = N_1$ we obtain

$$\begin{aligned}
\sup_{x\in S} |f(x)| &= \sup_{x\in S} |f_n(x) - f_n(x) + f(x)| \\
&\leq \sup_{x\in S} \big(|f_n(x)| + |f_n(x) - f(x)|\big) \\
&\leq \sup_{x\in S} |f_n(x)| + \sup_{x\in S} |f_n(x) - f(x)| \\
&\leq \sup_{x\in S} |f_n(x)| + 1.
\end{aligned}$$

Since f_n is bounded, this estimate implies that f is also bounded. □

Recall that $C[a, b]$ is the space of continuous functions $[a, b] \to \mathbb{R}$ equipped with the sup metric.

Corollary 6.10. *$C[a, b]$ is complete.*

Proof. Since continuous functions on $[a, b]$ are bounded, Theorem 6.9 implies that any Cauchy sequence of continuous functions (f_n) converges uniformly to a bounded function f on $[a, b]$, and we only need to prove that the function f is continuous.

Let $\alpha \in [a, b]$ and $\varepsilon > 0$ be arbitrary. We must show that there exists $\delta > 0$ such that $|f(x) - f(\alpha)| < \varepsilon$ whenever $|x - \alpha| < \delta$. We have

$$\begin{aligned} |f(x) - f(\alpha)| &= |f(x) - f_n(x) + f_n(x) - f_n(\alpha) + f_n(\alpha) - f(\alpha)| \\ &\le |f(x) - f_n(x)| + |f_n(x) - f_n(\alpha)| + |f_n(\alpha) - f(\alpha)|. \end{aligned}$$

Since $f_n \to f$ in $C[a, b]$, we can choose n such that $d_\infty(f_n, f) < \varepsilon/3$, where d_∞ is the sup metric on $C[a, b]$. For this choice of n, we have $|f(x) - f_n(x)| < \varepsilon/3$ and $|f(\alpha) - f_n(\alpha)| \le \varepsilon/3$. Since the function f_n is continuous, there exists $\delta > 0$ such that $|f_n(x) - f_n(\alpha)| < \varepsilon/3$ whenever $|x - y| < \delta$. Therefore

$$|f(x) - f(\alpha)| \le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

whenever $|x - \alpha| \le \delta$. □

6.3 Contractions

Definition 6.11. A map f from a metric space (X, ϱ) to itself is called a *contraction* if $\varrho(f(x), f(y)) \le c\varrho(x, y)$ for some $0 \le c < 1$ and all $x, y \in X$.

Lemma 6.12. *Let f be a mapping of metric space to itself. If f is a contraction then f is continuous.*

Proof. Suppose $\alpha \in X$ and $\varepsilon > 0$ are arbitrary. For any x with $\varrho(x, \alpha) < \varepsilon$ we have $\varrho(f(x), f(\alpha)) \le c\,\varrho(x, \alpha) < \varepsilon$. So the definition of continuity is met with $\delta = \varepsilon$. □

Theorem 6.13 (The Contraction Mapping Theorem)**.** *If f is a contraction on a complete metric space then the equation $f(x) = x$ has a unique solution x and, for any $x_0 \in X$, the sequence x_n defined by $x_n = f(x_{n-1})$, for all $n > 0$, converges to x.*

Proof. Let $r_0 = \varrho(x_0, x_1)$. Then

$$\varrho(x_k, x_{k+1}) = \varrho(f(x_{k-1}), f(x_k)) \le c\,\varrho(x_{k-1}, x_k),$$

for $k > 0$, and so, by induction on k, $\varrho(x_k, x_{k+1}) \le c^k r_0$. Let $n < m$. Then

$$\begin{aligned}\varrho(x_n, x_m) &\le \varrho(x_n, x_{n+1}) + \varrho(x_{n+1}, x_{n+2}) + \varrho(x_{m-1}, x_m)\\ &\le c^n r_0 + c^{n+1} r_0 + \cdots + c^{m-1} r_0\\ &\le c^n r_0 (1 + c + c^2 + \cdots)\\ &= c^n (1-c)^{-1} r_0. \end{aligned} \tag{5}$$

Since $c < 1$, the final expression can be made arbitrarily small by choosing n large. This implies that x_n is a Cauchy sequence. Since our metric space is complete, x_n converges to a limit x.

In view of Lemma 6.12, we have $f(x_n) \to f(x)$ as $n \to \infty$. At the same time, $f(x_n) = x_{n+1} \to x$ as $n \to \infty$. So, by looking at the limit of the convergent sequence $f(x_n)$ in two different ways, we conclude that $f(x) = x$, as required.

If y is another solution of the equation $f(y) = y$ then

$$0 = \varrho(f(x), f(y)) - \varrho(x, y) \le c\,\varrho(x, y) - \varrho(x, y) = (c-1)\varrho(x, y)$$

and, consequently, $\varrho(x, y) = 0$. This implies that x is the only solution of the equation $f(x) = x$. □

Theorem 6.13 allows one to construct an approximate solution to an equation of the form $f(x) = x$ by choosing an arbitrary element $x_0 \in X$ and evaluating $x_k = f^k(x)$ for sufficiently large m. This is called the *method of successive approximations.*

Corollary 6.14 (error estimate)**.** *Under the conditions of Theorem 6.13 we have*

$$\varrho(x_n, x) \le c^n (1-c)^{-1} \varrho(x_0, x_1), \quad \textit{for all } x_0 \in X \textit{ and all } n = 0, 1, 2, \ldots .$$

Proof. We know that x_m converges to x. Let $\varepsilon > 0$ and choose $m \ge n$ such that $\varrho(x_m, x) < \varepsilon$. Then

$$\begin{aligned}\varrho(x_n, x) &\le \varrho(x_n, x_m) + \varrho(x_m, x)\\ &\le c^n (1-c)^{-1} \varrho(x_0, x_1) + \varepsilon,\end{aligned}$$

by (5). But $\varepsilon > 0$ is arbitrary, and the result follows. □

Example 6.15. Consider the function $f(x) = x/2 + 1/x$ defined on $[1, \infty)$. The function f maps $[1, \infty)$ into itself. [Check this.] It is also a contraction (with respect to the usual metric):

$$|f(x) - f(y)| = \big|(x/2 + 1/x) - (y/2 + 1/y)\big| = \big|(x-y)(1/2 - 1/xy)\big| \le |x-y|/2.$$

Furthermore, $[1,\infty)$ is complete. [Check this.] Thus the sequence $f(1) = 1.5$, $f^2(1) = 1.4167-$, $f^3(1) = 1.4142+$ converges to a unique limit $x \in [1,\infty)$. Since $x = \sqrt{2}$ satisfies the equation $f(x) = x$, that limit is in fact $\sqrt{2}$. Since $c = \frac{1}{2}$ and $\varrho(1, f(1)) = \frac{1}{2}$, the error estimate from Corollary 6.14 is 2^{-n}. If we had started with an initial value other than 1 we would still have converged to the same limit.

Remark 6.16. For a function $f : \mathbb{R} \to \mathbb{R}$ the inequality $|f(x) - f(y)| \leq c\,|x - y|$, (with some $c > 0$) is called a *Lipschitz condition.* If f is continuously differentiable on $[a, b]$ then, by the mean value theorem, f satisfies the Lipschitz condition with $c = \sup\{|f'(x)| : x \in [a, b]\}$.

Example 6.17. Suppose we are looking for a solution to the differential equation

$$\frac{d\varphi}{dx} = \alpha\,\varphi(x), \quad \varphi(0) = 1,$$

in $C[0, b]$, i.e., among continuous functions $\varphi(x)$ on the closed interval $[0, b]$. (Of course, this is artificial, as we know the answer immediately!) Recast the above as an integral equation

$$\varphi(x) = \varphi(0) + \alpha \int_0^x \varphi(\xi)\,d\xi = 1 + \alpha \int_0^x \varphi(\xi)\,d\xi. \tag{6}$$

(This is legitimate by the Fundamental Theorem of calculus.) Now define the map $T : C[0, b] \to C[0, b]$ as follows:

$$(T(\varphi))(x) = 1 + \alpha \int_0^x \varphi(\xi)\,d\xi.$$

(That is to say: $T(\varphi)$ is a function whose value at $x \in [0, b]$ is given by the rhs of the equation.) We are using here the fact that a continuous function is integrable, and that the integral in the definition of T is a continuous function of x.) Then (6) is equivalent to the equation $T(\varphi) = \varphi$.

Now, we know (Corollary 6.10) that $C[0, b]$ is a complete metric space. So, if T is a contraction, then $T(\varphi) = \varphi$ has a unique solution, which we can approximate using the method of successive approximations. To this end, consider the following sequence of inequalities:

$$\begin{aligned}
d_\infty(T(\varphi), T(\psi)) &= \sup_{x\in[0,b]} |T(\varphi)(x) - T(\psi)(x)| \\
&= \sup_{x\in[0,b]} \left|\alpha \int_0^x \varphi(\xi) - \psi(\xi)\, d\xi\right| \\
&\le \sup_{x\in[0,b]} |\alpha| \int_0^x |\varphi(\xi) - \psi(\xi)|\, d\xi \\
&= |\alpha| \int_0^b |\varphi(\xi) - \psi(\xi)|\, d\xi \\
&\le b\,|\alpha| \sup_{\xi\in[0,b]} |\varphi(\xi) - \psi(\xi)| \\
&= b\,|\alpha|\, d_\infty(\varphi, \psi).
\end{aligned}$$

So T is a contraction, provided $b < |\alpha|^{-1}$.

How does this work in practice? Well, let's set φ_0 to be the constant function 0 (the choice is not critical), and define φ_n for $n > 0$ by $\varphi_n = T(\varphi_{n-1})$. Then (simple calculus):

$$\begin{aligned}
&\varphi_0(x) = 0, \\
&\varphi_1(x) = 1, \\
&\varphi_2(x) = 1 + \alpha x, \\
&\varphi_3(x) = 1 + \alpha x + (\alpha x)^2/2!\,, \\
&\varphi_4(x) = 1 + \alpha x + (\alpha x)^2/2! + (\alpha x)^3/3!\,,
\end{aligned}$$

etc. [Check this.] We can see φ_n tending uniformly to $\exp(\alpha x)$ as expected, and it even does so outside the range $[0, b]$ guaranteed by the contraction mapping theorem.

Of course, this is a toy example, but the same process can be used to show existence and uniqueness of solutions to differential equations under quite general conditions, and to provide an effective procedure for computing approximate solutions.

Let g be a real-valued function defined on an open domain $\Omega \subseteq \mathbb{R}^2$. Consider the ordinary (non-linear) differential equation

$$\frac{d\varphi}{dx} = g(x, \varphi(x)) \tag{7}$$

with the initial condition $\varphi(x_0) = \varphi_0$, where x takes values in $\mathbb{R}$, φ is a function of x and φ_0 is some constant.

Theorem 6.18 (Picard's Theorem). *Let* $(x_0, \varphi_0) \in \Omega$ *and* g *be a continuous function satisfying the Lipschitz condition*

$$|g(x, y_1) - g(x, y_2)| \leq c\,|y_1 - y_2|,$$

where c *is some constant. Then the equation (7) with the initial condition* $\varphi(x_0) = \varphi_0$ *has a unique solution on some interval* $[x_0 - \delta, x_0 + \delta]$.

Proof. This is beyond the scope of the course. However, the basic ideas are all in Example 6.17, and the various steps can be justified by reference to results from D&IA. The proof provides an explicit procedure for producing approximations to the solution φ. □