4 Open sets and closed sets

Throughout this section, (X, ϱ) is a metric space.

Definition 4.1. A set $A \subseteq X$ is *open* if it contains an open ball about each of its points. That is, for all $x \in A$, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$.

Lemma 4.2. *An open ball in a metric space (X, ϱ) is an open set.*

Proof. If $x \in B_r(\alpha)$ then $\varrho(x, \alpha) = r - \varepsilon$ where $\varepsilon > 0$. If $y \in B_\varepsilon(x)$ then $\varrho(x, y) < \varepsilon$ and, by the triangle inequality,

$$\varrho(\alpha, y) \leq \varrho(\alpha, x) + \varrho(x, y) < r - \varepsilon + \varepsilon = r.$$

This implies that $y \in B_r(\alpha)$ for all $y \in B_\varepsilon(x)$, that is, $B_\varepsilon(x) \subseteq B_r(\alpha)$. $\qquad\square$

Theorem 4.3. *If (X, ϱ) is a metric space then*

1. *the whole space X and the empty set $\emptyset$ are both open,*

2. *the union of any collection of open subsets of X is open,*

3. *the intersection of any finite collection of open subsets of X is open.*

Proof. (1) The whole space is open because it contains all open balls, and the empty set is open because it does not contain any points.

(2) Suppose $\{A_i : i \in I\}$ is a collection of open sets, indexed by I, and let $A = \bigcup_{i \in I} A_i$. Let $x \in A$ be arbitrary. Then x belongs to at least one of the sets A_i. Since this set is open, it contains an open ball about x; clearly, this ball lies in A. But $x \in A$ was chosen arbitrarily, and so A meets the definition of an open set.

(3) Suppose $A_1, A_2, \ldots, A_n$ are open sets and let $A = \bigcap_{i=1}^n A_i$. Let $x \in A$ be arbitrary. Then $x \in A_i$ for every $i = 1, \ldots, n$. For each i, since A_i is open, there exists $r_i > 0$ such that $B_{r_i}(x) \subseteq A_i$. Let $r = \min\{r_1, r_2, \ldots, r_n\}$. Then $B_r(x) \subseteq B_{r_i}(x) \subseteq A_i$ for all $i = 1, \ldots, n$, and hence $B_r(x) \subseteq A$. But $x \in A$ was chosen arbitrarily, and hence A is an open set. $\qquad\square$

An infinite intersection of open sets is not necessarily open.

Example 4.4. Let A_n be the open intervals $(-1/n, 1/n)$ in $\mathbb{R}$. Then A_n are open sets but the intersection $\bigcap_{n=1}^\infty A_n = \{0\}$ is not open.

Remark 4.5. In this course, we introduced the notion of metric, and then used that notion to define open sets. But it is possible to define the open sets directly to be any collection of subsets of X satisfying conclusions (1)–(3) of Theorem 4.3. This leads to the more general notion of *topological space*. Not every topological space is a metric space: there are collections of open sets satisfying (1)–(3) that do not arise from a metric on X.

Lemma 4.6. *A set is open if and only if it is equal to the union of a collection of open balls.*

Proof. According to Theorem 4.3(2) the union of any collection of open balls is open. On the other hand, if A is open then for every point $x \in A$ there exists a ball $B(x)$ about x lying in A. We have $A = \bigcup_{x \in A} B(x)$. Indeed, the union $\bigcup_{x \in A} B(x)$ is a subset of A because every ball $B(x)$ is a subset of A, and the union contains every point $x \in A$ because $x \in B(x)$. $\qquad\square$

Definition 4.7. The *interior* of a set A is the union of all open sets contained in A, that is, the maximal open set contained in A. The interior of A is denoted by $\text{int}(A)$.

The definion is legitimate because of Theorem 4.3(2).

Definition 4.8. If $A \subseteq X$ then $\mathcal{C}(A) = X \setminus A$ denotes the complement of the set A in X, that is, the set of all points $x \in X$ which do not belong to A.

Definition 4.9. A set is $A \subseteq X$ is *closed* iff its complement $\mathcal{C}(X)$ is open.

Lemma 4.10. *A closed ball in a metric space (X, ϱ) is a closed set.*

Proof. Consider the closed ball $B_r[\alpha]$. We need to show that $\mathcal{C}(B_r[\alpha])$ is open. Suppose x is any point in $\mathcal{C}(B_r[\alpha])$. Since x is not in $B_r[\alpha]$, it must be the case that $\varrho(\alpha, x) > r$. Choose $\varepsilon > 0$ such that $\varrho(\alpha, x) > r + \varepsilon$. We claim that $B_r[\alpha] \cap B_\varepsilon(x) = \emptyset$. Suppose to the contrary that $B_r[\alpha]$ and $B_\varepsilon(x)$ have a point z in common. Then, by the triangle inequality, $\varrho(\alpha, x) < \varrho(\alpha, z) + \varrho(z, x) < r + \varepsilon$, which contradicts the choice of ε. We have shown that every point $x \in \mathcal{C}(A)$ is the centre of ball $B_\varepsilon(x)$ completely contained in $\mathcal{C}(A)$, and hence $\mathcal{C}(A)$ is open, as required. $\qquad\square$

Definition 4.11. A point $x \in X$ is called a *limit* point of a set A if every ball about x contains a point of A distinct from x. The set of limit points of A is denoted A'.

Other terms for "limit point" are point of accumulation or cluster point.

Theorem 4.12. *A set is closed if it contains all its limit points.*

Proof. Suppose A is closed. Then, by definition, the complement $\mathcal{C}(A) = X \setminus A$ is open. Take any point $x \in \mathcal{C}(A)$ lying outside A. Since $\mathcal{C}(A)$ is open, there is an $\varepsilon > 0$ such that the ball $B_\varepsilon(x) \subseteq \mathcal{C}(A)$ lies entirely within $\mathcal{C}(A)$. Clearly, $B_\varepsilon(x) \cap A = \emptyset$, demonstrating that x is not a limit point of A. But the choice of $x \in \mathcal{C}(A)$ was arbitrary, and so A must contain all its limit points.

Suppose A contains all its limit points. Let $x \in \mathcal{C}(A)$ be arbitrary. Since x is not a limit point, there is an $\varepsilon > 0$ such that the ball $B_\varepsilon(x) \cap A = \emptyset$. Thus $\mathcal{C}(A)$ is open and A is closed. $\qquad\square$

Theorem 4.13. *In a metric space (X, ϱ)*

1. *the whole space X and the empty set $\emptyset$ are both closed,*

2. *the intersection of any collection of closed sets is closed,*

3. *the union of any finite collection of closed sets is closed.*

Proof. The theorem follows from Theorem 4.3 and the definition of closed set.

(1) $\mathcal{C}(X) = \emptyset$ and $\mathcal{C}(\emptyset) = X$.

(2) Suppose $\{A_i : i \in I\}$ is a collection of sets, indexed by I, and let $A = \bigcap_{i \in I} A_i$. Then

$$\mathcal{C}(A) = \mathcal{C}\Big(\bigcap_{i \in I} A_i\Big) = \bigcup_{i \in I} \mathcal{C}(A_i).$$

The r.h.s. is a union of a collection of open sets and hence open. Thus, by definition, A is closed.

(3) Suppose $A = \bigcup_{n=1}^{k} A_i$ is a finite union of closed sets. then

$$\mathcal{C}(A) = \mathcal{C}\Big(\bigcup_{i=1}^{k} A_i\Big) = \bigcap_{i=1}^{n} \mathcal{C}(A_i).$$

The r.h.s. is a finite intersection of open sets and hence open. Thus, by definition, A is closed. $\qquad\square$

Definition 4.14. The *closure* of a set A is the intersection of all closed sets containing A, that is, the minimal closed set containing A. The closure is denoted by $\mathrm{cl}(A)$ or $\overline{A}$.

The definition is legitimate because of Theorem 4.13(2). Clearly, $\mathrm{int}(A) \subseteq A \subseteq \overline{A}$.

Theorem 4.15. $\overline{A} = A \cup A'$.

The proof is omitted.