A faster and simpler strongly polynomial algorithm for generalized flow maximization

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Joint work with

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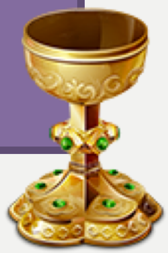
Linear Programming
& strongly polynomial algorithms

Is there a strongly polynomial algorithm for Linear Programming?



$$\min c^\top x$$
$$Ax = b$$
$$x \geq 0$$

$$A \in \mathbb{Q}^{m \times n}$$

- Strongly polynomial
  - poly(n, m) elementary arithmetic operations and comparisons
  - PSPACE: The numbers do not get too large during the computations
- Weakly polynomial
  - Running time depends on total encoding length L
  - Ellipsoid algorithm, interior point methods

Linear Programming
& strongly polynomial algorithms

- Network flow problems:
  - Maximum flow: Edmonds-Karp-Dinitz '70-72
  - Min-cost flow: Tardos '85

- Combinatorial LP's: integer matrix A.
  - $\mathrm{poly}(n, m, \log \Delta)$ for $A \in \mathbb{Z}^{m \times n}$ with max. subdeterminant Δ: Tardos '86, Vavasis-Ye '96

LP's with at most two nonzero entries per column in the constraint matrix

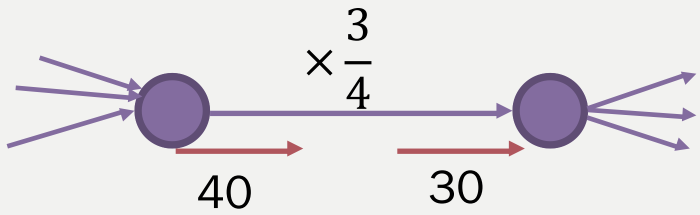
$$\min c^\top x$$
$$Ax = b$$
$$x \geq 0$$

$$A = \begin{bmatrix} * & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & * & 0 & * \\ 0 & 0 & * & 0 \end{bmatrix}$$

- Dual feasibility
  solving a system of inequalities with ≤ 2 variables per inequality: Megiddo '83
- Primal feasibility
  reduces to *max. generalized flow:* V '14, this paper
- Optimization problem
  equivalent to *min-cost generalized flow*: open
- Three nonzeros
  equivalent to general LP

Generalized flow maximization

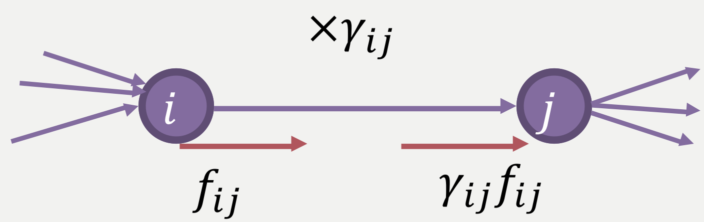
- Network flow model with gain factors



- Gain factors can model:
  - *physical changes (e.g. leakage, theft)*
  - *conversions between different types of entities (e.g. currencies, products)*
- Objective: maximize the flow amount reaching the sink t.

Generalized flow maximization

- Network flow model with gain factors



$$\begin{array}{cc} & ij \\ \begin{array}{c} i \\ \\ j \\ \\ \end{array} & \left[\begin{array}{c} -1 \\ 0 \\ \gamma_{ij} \\ 0 \end{array}\right] \end{array}$$

- Flow balance at node i

$$\nabla f_i = \sum_{j \in \delta^{in}(i)} \gamma_{ji} f_{ji} - \sum_{j \in \delta^{out}(i)} f_{ij}$$

$$\max \nabla f_t$$

s.t. $\nabla f_i \geq b_i \quad \forall i \in V - t$

$$f \geq 0$$

Generalized flow maximization

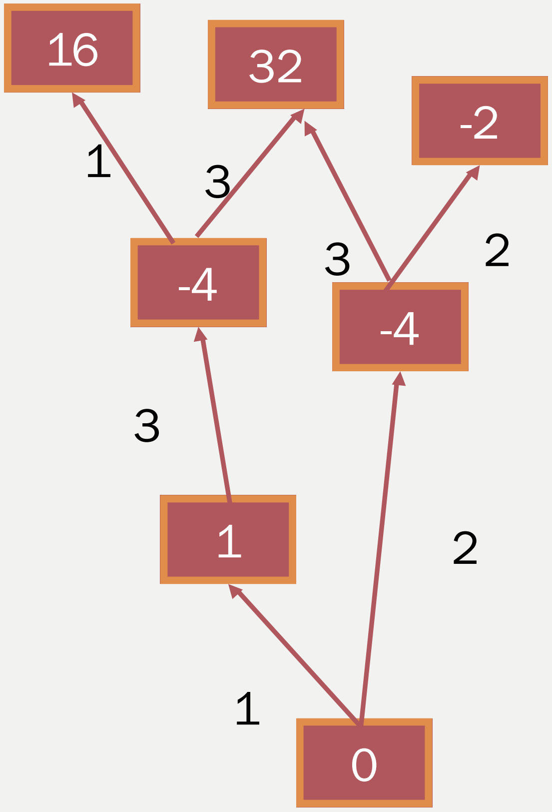
- n: #nodes, m: #arcs, B: largest integer in input data
- Kantorovich ’39, Dantzig ’62: model introduced
- Goldberg, Plotkin & Tardos ’91: first polynomial combinatorial algorithm
- Several weakly polynomial algorithms: Cohen & Megiddo ’94, Goldfarb, Jin ’96, Goldfarb, Jin & Orlin ’97, Tardos & Wayne ’98, Wayne ’02, Daitch & Spielman ’08, Restrepo & Williamson ’09, V. ’12, ...
- Best previous running times:
  - Vaidya ’89: $O(m^{1.5}n^2 \log B)$
  - Radzik ’04: $O\big((m + n\log n)mn\log B\big)$
- V ’14: $O(m^3n^2)$
- Olver-V ’17: $O\left((m + n\log n)mn\log\left(\frac{n^2}{m}\right)\right)$

Detour:
Net Present Value Problem

- Critical Path Problem with time discounts
- Introduced by Russell ’70
- Application of generalized flows
- Strongly polynomial computability becomes a crucial issue

Net Present Value Problem

- Jobs J, precedence constraints $\prec$
- Processing time $p_j \in \mathbb{N}$, profit/loss $b_j \in \mathbb{Z}$ for each $j \in J$
- Discount factor $\rho > 1$



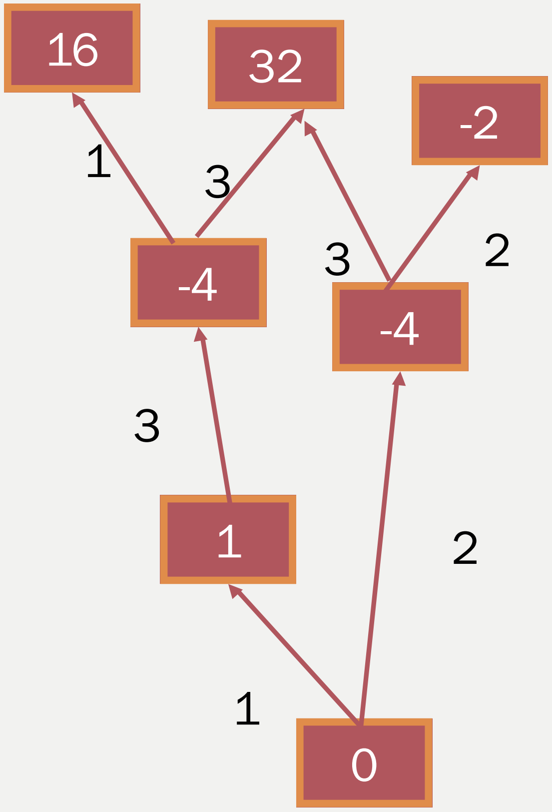
$$\max \sum_{j \in J} b_j \rho^{-c_j}$$
$$\text{s.t. } c_j \geq c_i + p_j \quad \forall i \prec j$$
$$c_t = 0, c \geq 0$$

Net Present Value Problem

$$\max \sum_{j \in J} b_j \rho^{-c_j}$$
$$\text{s.t. } c_j \geq c_i + p_j \quad \forall i \prec j$$
$$c_t = 0, c \geq 0$$

- $\mu_j := \rho^{c_j}, \gamma_{ij} := \rho^{p_j}$

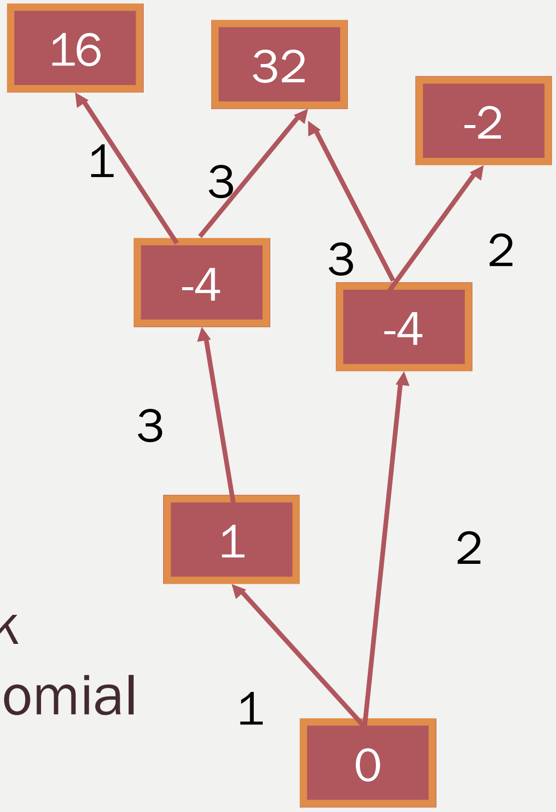
$$\max \sum_{j \in J} \frac{b_j}{\mu_j}$$
$$\text{s.t. } \mu_j \geq \gamma_{ij} \mu_i \quad \forall i \prec j$$
$$\mu_t = 1, \mu > 0$$



Net Present Value Problem

- $\mu_j := \rho^{c_j}, \gamma_{ij} := \rho^{p_j}$

$$\max \sum_{j \in J} \frac{b_j}{\mu_j}$$
$$\text{s.t.} \quad \mu_j \geq \gamma_{ij}\mu_i \quad \forall i \prec j$$
$$\mu_t = 1, \mu > 0$$



- Dual of generalized flow maximization
- Weakly polynomial algorithms cannot work
- A variant of our algorithm is strongly polynomial for this problem (at least for $\rho \in \mathbb{N}$): Correa, Olver, Schulz, V.

Generalized flow maximization

- Flow balance at node i

$$\nabla f_i = \sum_{j \in \delta^{in}(i)} \gamma_{ji} f_{ji} - \sum_{j \in \delta^{out}(i)} f_{ij}$$

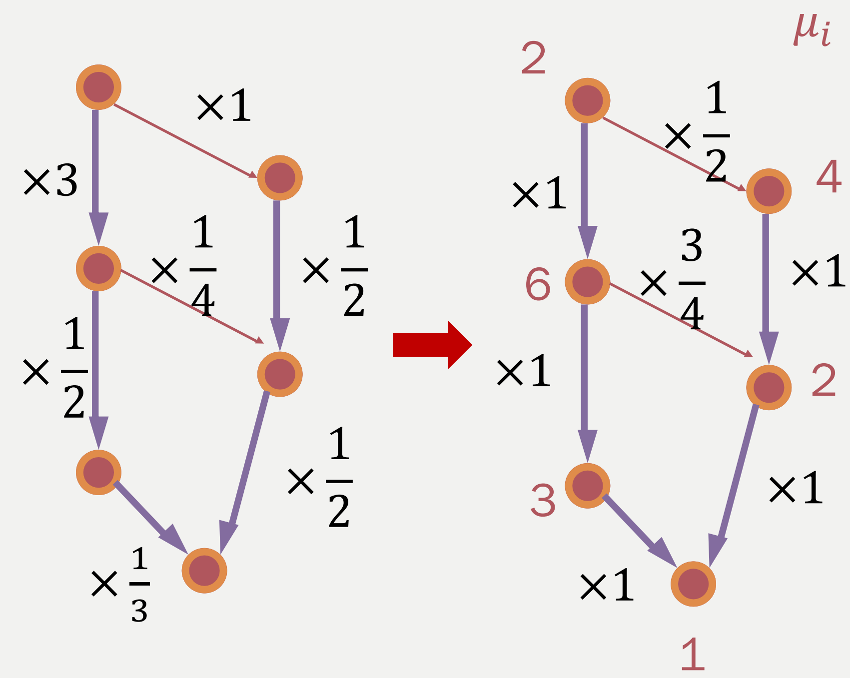
$$\begin{aligned} & \max \nabla f_t \\ \text{s.t.} \quad & \nabla f_i \geq b_i \quad \forall i \in V - t \\ & f \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} & \max \mu_t \sum_{j \in J} \frac{b_j}{\mu_j} \\ \text{s.t.} \quad & \mu_j \geq \gamma_{ij} \mu_i \quad \forall ij \in E \\ & \mu > 0 \end{aligned}$$

Labellings and duality

- Change measurement unit
- $\mu : V \to \mathbb{R}_{>0}$
- $f_{ij}^{\mu} := \frac{f_{ij}}{\mu_i}$, $\quad \gamma_{ij}^{\mu} := \frac{\gamma_{ij}\mu_i}{\mu_j}$,
  $b_i^{\mu} := \frac{b_i}{\mu_i}$, $\quad \nabla f_i^{\mu} := \frac{\nabla f_i}{\mu_i}$
- $ij \in E$ is tight if $\gamma_{ij}^{\mu} = 1$
- (f, μ) is a fitting pair if
  - $\gamma_{ij}^{\mu} \leq 1 \quad \forall ij \in E$
    dual feasibility
  - $\gamma_{ij}^{\mu} = 1$, if $f_{ij} > 0$
    complementary slackness



Dual

$$\max \mu_t \sum_{j \in J} b_j^{\mu}$$

$$\text{s.t.} \quad \gamma_{ij}^{\mu} \leq 1 \quad \forall ij \in E$$

$$\mu > 0$$

Main progress: abundant arcs

- Main goal: find a dual optimal solution μ^*.
- Primal optimal solution: max normal flow computation on tight arcs w.r.t. μ^*.
- Main progress:
  - *find an abundant arc a that must be tight for every optimal dual solution*
  - *contract a and recurse.*
- Standard technique for minimum cost flows: Orlin '93
  - *also used in V. '14*

Main progress: abundant arcs

- Feasible flow: $\nabla x_i \geq b_i \quad \forall i \in V - t$
- Total relabeled excess

$$Ex(x,\mu) := \sum_{i \in V-t} (\nabla x_i^\mu - b_i^\mu)$$

LEMMA: If x is a feasible flow and (x,μ) is a fitting pair, and

$$x_a^\mu > Ex(x,\mu),$$

Then a is an abundant arc.

- Proof: standard flow decomposition technique.

Main progress: contractible arcs

Assume the following hold for flow f, and labelling μ:

- (f, μ) is a fitting pair (complementary slackness)
- $\nabla f_i^\mu \geq b_i^\mu \quad \forall i \in V - t$ (feasibility)
- $\nabla f_i^\mu \leq b_i^\mu + 2$ (small excess)

Then

- every arc a with $f_a^\mu > 2n \geq Ex(x, \mu)$ is contractible;
- if $\left|b_i^\mu\right| > 2n^2$, then there exists a contractible arc incident to node i.

Main progress: contractible arcs

Assume the following hold for flow f, and labelling μ:

- (f, μ) is a fitting pair (complementary slackness)
- $\nabla f_i^\mu \geq b_i^\mu \quad \forall i \in V - t \quad$ (feasibility)
- $\nabla f_i^\mu \leq b_i^\mu + 2$ (small excess)

- Problem: difficult to have both fitting pairs and feasibility in an augmenting path framework
- Many previous algorithms relax the fitting pair notion
- New idea: relax feasibility

Algorithmic setup

Maintain a flow f, and labelling μ:

- (f, μ) is a fitting pair (complementary slackness)
- ~~$\nabla f_i^\mu \geq b_i^\mu \ \forall i \in V - t$ (feasibility)~~
- $\nabla f_i^\mu \leq b_i^\mu + 2$ (small excess)
- μ is a safe labelling: there exists a primal feasible solution on tight arcs for μ.
  Can be characterized by a simple cut condition

LEMMA: If (f, μ) is a fitting pair, and μ is <u>safe,</u> then there exists a flow x such that (x, μ) is a fitting pair and

$$b_i \leq \nabla x_i \leq \max(b_i, \nabla f_i) \ \forall i \in V - t$$

- f^μ is integral and small ($f_a^\mu \leq 2n$ for all a) 🙂

Work with two flows instead of one!

LEMMA: If (f, μ) is a fitting pair, and μ is **safe**, then there exists a flow x such that (x, μ) is a fitting pair and

$$b_i \leq \nabla x_i \quad \leq \max(b_i, \nabla f_i) \;\; \forall i \in V - t$$

- If $\left|b_i^\mu\right| > 2n^2$, then compute such an x by a flow feasibility algorithm.

This satisfies:

- (x, μ) is a fitting pair (complementary slackness)
- $\nabla x_i^\mu \geq b_i^\mu \quad \forall i \in V - t$ (feasibility)
- $\mathrm{Ex}(\mathrm{x}, \mu) \leq 2n$ (small excess)
- There is an arc a with $x_a^\mu > 2n$ and this is contractible.

Main step

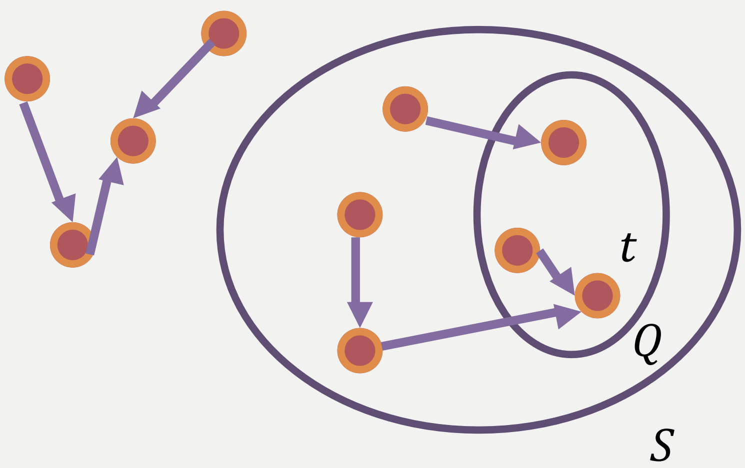
multiplicative Dijkstra-type update

- $Q \coloneqq \{t\} \cup \{j \in V : \nabla f_j^\mu < b_j^\mu\}$ – "sink set"
- Try send 1 unit of relabelled flow from a node j with $\nabla f_j^\mu \geq b_j^\mu + 1$ to a node in Q on a tight path.
- S: set of nodes that can reach Q on a tight path

Update for certain $\alpha > 1$

$$\mu_i := \begin{cases} \frac{\mu_i}{\alpha}, & if\ i \in S \\ \mu_i, & if\ i \in V \setminus S \end{cases}$$

$$f_a := \begin{cases} \frac{f_a}{\alpha}, & if\ a \in E[S] \\ f_a, & otherwise \end{cases}$$



Main step
multiplicative Dijkstra-type update

Update for certain $\alpha > 1$

$$\mu_i := \begin{cases} \dfrac{\mu_i}{\alpha}, & if\ i \in S \\ \mu_i, & if\ i \in V \setminus S \end{cases}$$

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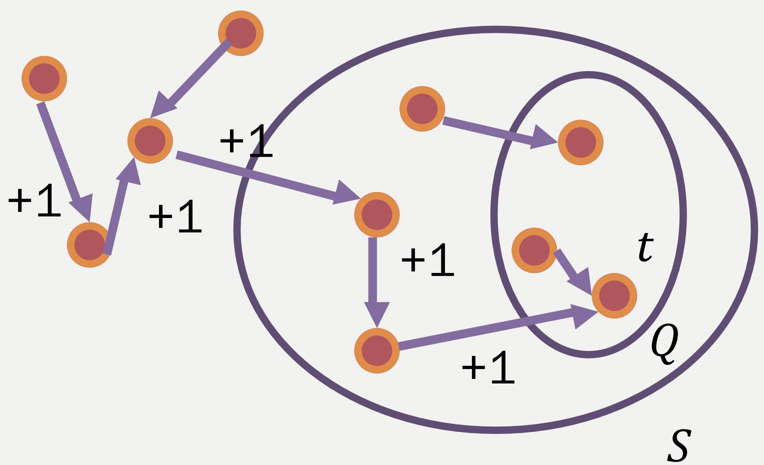

- The relabeled flow f^μ does not change.
- Arcs entering S may become tight.
- Safety of labelling μ is maintained.

Analysis key idea

Update for certain $\alpha > 1$

$$\mu_i := \begin{cases} \frac{\mu_i}{\alpha}, & if\ i \in S \\ \mu_i, & if\ i \in V \setminus S \end{cases}$$

$$f_a := \begin{cases} \frac{f_a}{\alpha}, & if\ a \in E[S] \\ f_a, & otherwise \end{cases}$$

- How does the excess of a node change?

$$\nabla f_i^\mu - b_i^\mu$$

- Can increase only if $i \in S$ and $b_i < 0$.

$$\Phi := \sum_{i:b_i<0} \nabla f_i^\mu - b_i^\mu \text{ and } \Psi := \sum_{i:b_i<0} |b_i^\mu|$$

- The two potentials increase together at label updates.
- Path augmentations almost always decrease Φ.

Comparison with V '14

V. '14	Current paper
$O(m^3n^2)$	$O\left((m+n\log n)mn\log\left(\frac{n^2}{m}\right)\right)$
Main progress by contracting arcs. Flow sent on tight paths.	
Δ-relaxation of (f,μ) fitting Feasible flow f Non-integral f^{μ}	(f,μ)-fitting pair Infeasible flow f, safe labelling μ Integral f^{μ}
Multiplicative potential argument	Additive potential argument
Additional cleanup step	-
Complicated numerical rounding	Simple numerical rounding

Conclusion

- Amortizing here and there, we get

$$O\left((m + n\log n)mn\log\left(\frac{n^2}{m}\right)\right)$$

- Also works for the Net Present Value problem for integer discount factor.
- Open questions:
  - *Net Present Value problem with rational discount factors.*
  - *Strongly polynomial algorithm for minimum cost generalized flows.*
  - *Strongly polynomial algorithm for LP.*

Thank you!