The Switch Chain on Perfect Matchings and the Recognition of Quasimonotone Graphs

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Motivation: Counting Perfect Matchings

Counting the number of perfect matchings in bipartite graphs is

- $\#\mathbb{P}$-complete [Valiant 1979]
- $\#\mathbb{P}$-complete for max degree 3 [Dagum and Luby 1992]
- $\#\mathbb{P}$-complete on chordal bip. [Okamoto, Uehara, Uno 2010]
- $\mathbb{FP}$ for planar graphs [Kasteleyn 1963]
- FPTRAS [Jerrum and Sinclair 1989, with Vigoda 2004]
- weighted problem known as computing the permanent

The switch chain

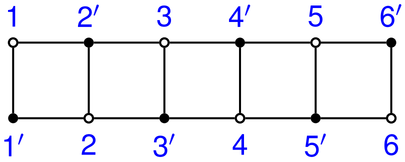
The switch chain

- proposed by Diaconis, Graham and Holmes 2001
- ergodic on chordal bipartite graphs
- rapidly mixing on monotone graphs [Dyer, Jerrum, M 2016]
- extended to non-bipartite graphs [Dyer, M 2017]
- not rapidly mixing on bi-convex graphs [Matthews 2008]

Monotone Graphs

A bipartite graph $G = (X, Y, E)$ is monotone if X and Y can be permuted such that the bi-adjacency matrix

- has the consecutive 1s property
- the boundaries between 0s and 1s are monotonically non-decreasing



	$1'$	$2'$	$3'$	$4'$	$5'$	$6'$
1	1	1	0	0	0	0
2	1	1	1	0	0	0
3	0	1	1	1	0	0
4	0	0	1	1	1	0
5	0	0	0	1	1	1
6	0	0	0	0	1	1

Monotone graphs are also know as

- bipartite permutation graphs [Spinrad, Brandstädt, and Stewart 1987]
- proper interval bigraphs [Hell and Huang 2004]

Bipartition

Let $G = (V, E)$, $L \subseteq V$ and $R = V \setminus L$. L, R is a bipartition of V.

The graph $G[L{:}R]$ is $(L, R, \{xy \in E : x \in L, y \in R\})$ is a bipartition of G.

For $\mathcal{C} \subseteq$ BIPARTITE let quasi-$\mathcal{C}$ be the class of all graphs G such that $G[L{:}R] \in \mathcal{C}$ for all bipartitions L, R of V.

Lemma

If $\mathcal{C} \subseteq$ BIPARTITE is a hereditary class that is closed under disjoint union then $\mathcal{C} =$ BIPARTITE $\cap$ quasi-$\mathcal{C}$.

Quasi-Classes and Pre-Graphs

For $H \in \textsc{Bipartite}$, a graph $G = (V, E)$ is a pre-H if there is a bipartition L, R of V such that $G[L{:}R] = H$.

Especially, a pre-hole is a pre-C_{2k} for $k \geq 3$.

Lemma

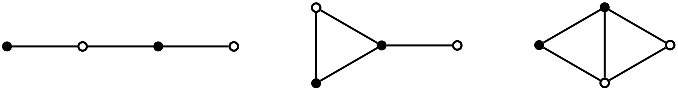
If $\mathcal{C} \subseteq \textsc{Bipartite}$ is characterised by a set $\mathcal{F}$ of forbidden induced subgraphs. Then quasi-$\mathcal{C}$ is characterised by the set of forbidden induced subgraphs pre-$\mathcal{F} = \{\text{pre-}H \mid H \in \mathcal{F}\}$.

Examples

The class quasi-BIPARTITE is the set of all graphs.

If $\mathcal{C}$ is the class of complete bipartite graphs then quasi-$\mathcal{C}$ is the class of complete graphs.

If $\mathcal{C}$ becomes the class of graphs for which every component is complete bipartite, then quasi-$\mathcal{C}$ is the class of graphs without P_4, paw or diamond.



More Examples

If $\mathcal{C}_d$ is the class of bipartite graphs with degree at most d then quasi-$\mathcal{C}_d$ is the class of all graphs with degree at most d.

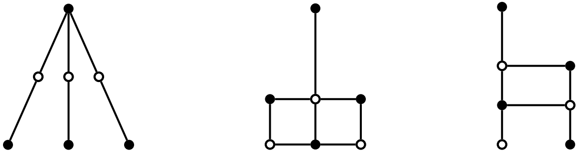
The quasi-class of linear forests contains all graphs with connected components that are either a path or an odd cycle.

OddChordal is the class of graphs in which every even cycle has an odd chord; quasi-ChordalBipartite $=$ OddChordal.

quasi-Monotone $=$ Quasimonotone.

Flaws and Preholes

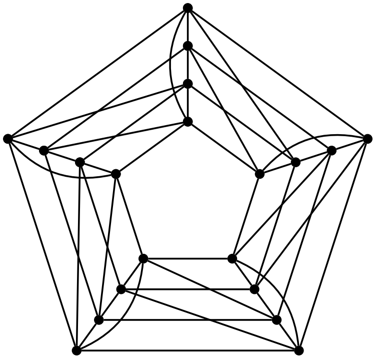
The bipartite minimal forbidden subgraphs of MONOTONE are all the even holes and three more graphs called tripod, stirrer and armchair [Köhler 1999].

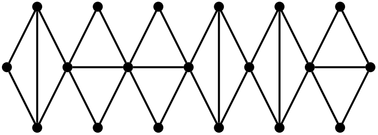


Quasimonotone graphs are characterised by the absence of preholes and flaws: pre-tripods, pre-stirrers and pre-armchairs.

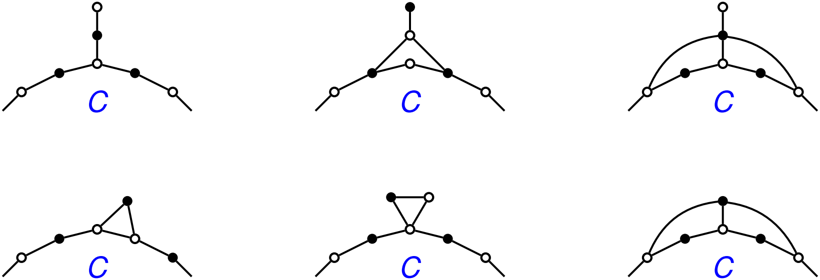
All monotone and all unit interval graphs are quasimonotone.

Two Quasimonotone Graphs





Holes in Flawless Graphs



First Steps

Given an input graph G:

- Is there a flaw in G?
- Is there a prehole of length 12 or less?
- Find a hole C in G in time $\mathcal{O}(m^2)$ [Nikolopoulos/Palios 2007]
- If C has vertices u and v with $d_C(u, v) < d_G(u, v)$ then shorten C.
- Distinguish the cases $|C| \geq 7$ and $|C| \leq 5$.

Long Hole and Triangle

Lemma

No quasimonotone graph with odd hole of size at least 7 contains a triangle sharing a vertex with that hole.

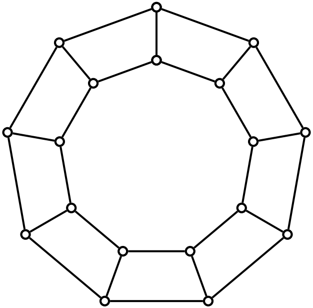
Lemma

No quasimonotone graph with odd hole of size at least 7 contains a triangle disjoint with that hole.

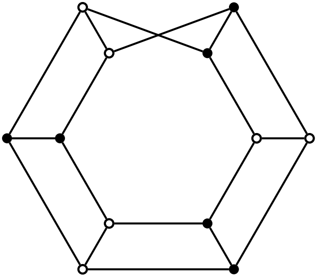
Prisms

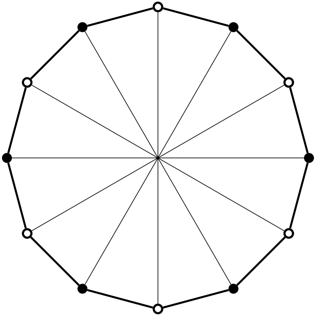
Lemma

Vertex-disjoint odd holes in a quasimonotone graph have the same length. If this length is ≥ 7 *then the holes induce a prism.*

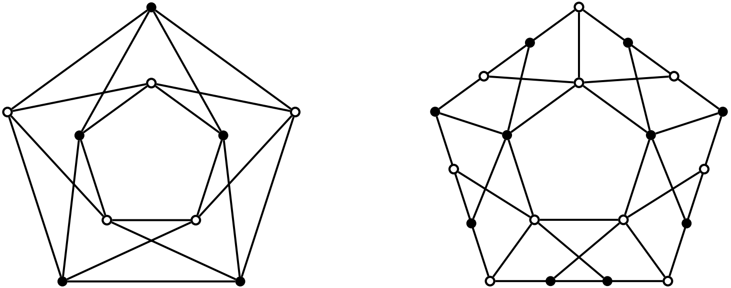


Möbius Ladder



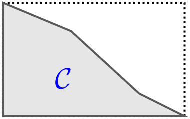


Crossover Preholes

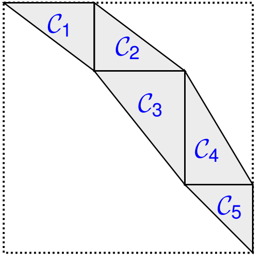


Decomposition into Chain Graphs

A bipartite graph (X, Y, E) is a chain graph if the inclusion of neighbourhoods defines linear orders on X and Y.

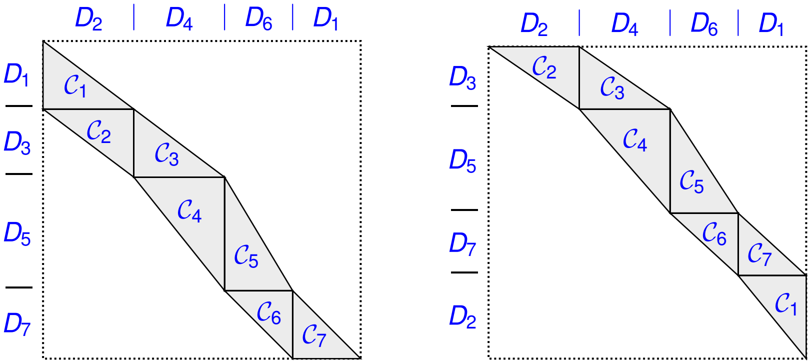


A monotone graph decomposes into chain graphs [Brandstädt and Lozin 2003].



Splitting

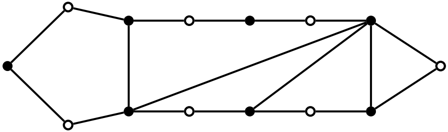
For a vertex v of G let $G_v = G - N[v]$.



Triangles and 5-Holes only

Lemma

Let C be a minimal prehole in a flawless graph G without odd holes ≥ 5. If C connects a 5-hole and a triangle or two 5-holes, then $|C| \leq 12$.



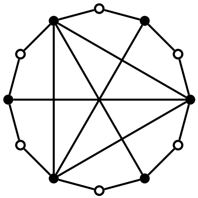
Triangles only

Let C be a minimal prehole in a flawless hole-free graph G.
A triangle in $G[C]$ will be called

- an interior triangle of C if it has no edge in common with C,
- a crossing triangle if it has one edge in common with C, and
- a cap of C if it has two edges in common with C.

Lemma

If C is a minimal prehole in a flawless graph with $|C| > 12$, then $G[C]$ has no interior or crossing triangles, and C is determined by two edge-disjoint caps.



Containment of Classes

$\Uparrow$ the switch chain is not ergodic

$\Downarrow$ the switch chain is ergodic

$\upuparrows$ counting perfect matchings remains $\#\mathbb{P}$-complete

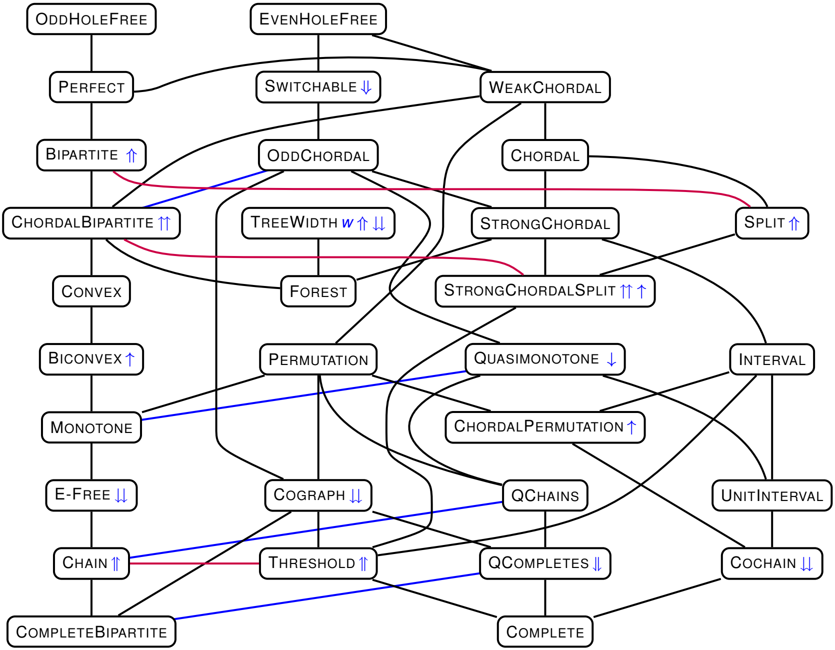
$\downdownarrows$ perfect matchings can be counted in polynomial time

$\uparrow$ the switch chain mixes slowly

$\downarrow$ the switch chain mixes rapidly

$\Uparrow$ contains graphs that are not P-stable

$\Downarrow$ all graphs in this class are P-stable.


OddHoleFree
EvenHoleFree
Perfect
Switchable ⇓
WeakChordal
Bipartite ⇑
OddChordal
Chordal
ChordalBipartite ⇈
TreeWidth w ⇑ ⇊
StrongChordal
Split ⇑
Convex
Forest
StrongChordalSplit ⇈ ↑
Biconvex ↑
Permutation
Quasimonotone ↓
Interval
Monotone
ChordalPermutation ↑
E-Free ⇊
Cograph ⇊
QChains
UnitInterval
Chain ⇑
Threshold ⇑
QCompletes ⇓
Cochain ⇊
CompleteBipartite
Complete