Martin Dyer:
A good friend and a great collaborator.

Coloring (Random) Hypergraphs

Alan Frieze

Carnegie Mellon University

Joint with Michael Anastos

Notation

We consider k-uniform hypergraphs with vertex set $V = [n] = \{1, 2, \dots, n\}$ and edges $E \subseteq \binom{V}{k}$.

Notation

We consider k-uniform hypergraphs with vertex set $V = [n] = \{1, 2, \ldots, n\}$ and edges $E \subseteq \binom{V}{k}$.

A proper q-coloring of $H = (V, E)$ is a map $\phi : V \to [k]$ such that no edge is monochromatic.

Notation

We consider k-uniform hypergraphs with vertex set $V = [n] = \{1, 2, \ldots, n\}$ and edges $E \subseteq \binom{V}{k}$.

A proper q-coloring of $H = (V, E)$ is a map $\phi : V \to [k]$ such that no edge is monochromatic.

The chromatic number $\chi(H)$ is the minimum value q such that H has a proper q-coloring.

Notation

We consider k-uniform hypergraphs with vertex set $V = [n] = \{1, 2, \ldots, n\}$ and edges $E \subseteq \binom{V}{k}$.

A proper q-coloring of $H = (V, E)$ is a map $\phi : V \to [k]$ such that no edge is monochromatic.

The chromatic number $\chi(H)$ is the minimum value q such that H has a proper q-coloring.

A hypergraph H is simple if $|e \cap f| \leq 1$ for all $e, f \in E$.

Notation

We consider k-uniform hypergraphs with vertex set $V = [n] = \{1, 2, \ldots, n\}$ and edges $E \subseteq \binom{V}{k}$.

A proper q-coloring of $H = (V, E)$ is a map $\phi : V \to [k]$ such that no edge is monochromatic.

The chromatic number $\chi(H)$ is the minimum value q such that H has a proper q-coloring.

A hypergraph H is simple if $|e \cap f| \leq 1$ for all $e, f \in E$.

The random hypergraphs $H_{n,m:k}$ (resp. $H_{n,p:k}$) have vertex set $[n]$ and m random edges from $\binom{V}{k}$ (resp. each edge in $\binom{V}{k}$ is included with probability p).

The chromatic number of random hypergraphs

The case of random graphs, $k = 2$, has been well-researched. For $k \geq 3$ we have

Theorem (Dyer, Frieze, Greenhill (2015))

Define $u_{k,q} = q^{k-1} \ln q$ *for integers* $k \geq 2$ *and* $q \geq 1$*. Suppose that* $k \geq 2$, $q \geq 1$*, and let* c *be a positive constant. Then for* $H = H_{n,cn;k}$,

1. *If* $c \geq u_{k,q}$ *then w.h.p.* $\chi(H) > q$.
2. *If* $k \geq 2$ *and* $\max\{k, q\} \geq 3$ *then there exists a constant* $c_{k,q} \in (u_{k,q-1}, u_{k,q})$ *such that if* $c < c_{k,q}$ *is a positive constant then w.h.p.* $\chi(H) \leq q$.

In particular this generalises an earlier result of Achlioptas and Naor [2005].

The chromatic number of random hypergraphs

The case of random graphs, $k = 2$, has been well-researched. For $k \geq 3$ we have

Theorem (Dyer, Frieze, Greenhill (2015))

Define $u_{k,q} = q^{k-1} \ln q$ for integers $k \geq 2$ and $q \geq 1$. Suppose that $k \geq 2$, $q \geq 1$, and let c be a positive constant. Then for $H = H_{n,cn;k}$,

1. *If $c \geq u_{k,q}$ then w.h.p. $\chi(H) > q$.*
2. *If $k \geq 2$ and $\max\{k, q\} \geq 3$ then there exists a constant $c_{k,q} \in (u_{k,q-1}, u_{k,q})$ such that if $c < c_{k,q}$ is a positive constant then w.h.p. $\chi(H) \leq q$.*

In particular this generalises an earlier result of Achlioptas and Naor [2005].

The theorem was later sharpened by Ayre, Coja-Oghlan and Greenhill [2018].

MCMC Algorithms for coloring (random) graphs

Given a graph G and $q \geq \chi(G)$, there is the problem of generating a (near) random proper coloring of G.

MCMC Algorithms for coloring (random) graphs

Given a graph G and $q \geq \chi(G)$, there is the problem of generating a (near) random proper coloring of G.

The first result in this area is due to Jerrum (1995) who showed that an MCMC approach works for $q \geq 2\Delta(G)$.

MCMC Algorithms for coloring (random) graphs

Given a graph G and $q \geq \chi(G)$, there is the problem of generating a (near) random proper coloring of G.

The first result in this area is due to Jerrum (1995) who showed that an MCMC approach works for $q \geq 2\Delta(G)$.

This was improved to $q \geq 1.76\Delta$ by Vigoda (2000).

MCMC Algorithms for coloring (random) graphs

Given a graph G and $q \geq \chi(G)$, there is the problem of generating a (near) random proper coloring of G.

The first result in this area is due to Jerrum (1995) who showed that an MCMC approach works for $q \geq 2\Delta(G)$.

This was improved to $q \geq 1.76\Delta$ by Vigoda (2000).

This was only recently improved to $q \geq (1.76 - \epsilon)\Delta$ by Chen and Moitra (2018) and by Delcourt, Perarnau and Postle (2018). Here $0 < \epsilon \leq 9.4 \times 10^{-5}$.

Given a graph G and $q \geq \chi(G)$, there is the problem of generating a (near) random proper coloring of G.

The first result in this area is due to Jerrum (1995) who showed that an MCMC approach works for $q \geq 2\Delta(G)$.

This was improved to $q \geq 1.76\Delta$ by Vigoda (2000).

This was only recently improved to $q \geq (1.76 - \epsilon)\Delta$ by Chen and Moitra (2018) and by Delcourt, Perarnau and Postle (2018). Here $0 < \epsilon \leq 9.4 \times 10^{-5}$.

For sparse random graphs, Δ is not a good measure of the number of colors needed.

Dyer, Flaxman, Frieze and Vigoda (2006) showed that w.h.p. $q = o(\Delta)$ suffices for the random graph $G_{n,p}, p = d/n$.

MCMC Algorithms for coloring (random) graphs

Dyer, Flaxman, Frieze and Vigoda (2006) showed that w.h.p. $q = o(\Delta)$ suffices for the random graph $G_{n,p}, p = d/n$.

After a series of improvements by Mossel and Sly; Efthymiou it has now been shown by Efthymiou, Hayes,Štefankovič and Vigoda that w.h.p. $q \approx 1.7632\ldots d$ is sufficient.

MCMC Algorithms for coloring (random) graphs

Dyer, Flaxman, Frieze and Vigoda (2006) showed that w.h.p. $q = o(\Delta)$ suffices for the random graph $G_{n,p}, p = d/n$.

After a series of improvements by Mossel and Sly; Efthymiou it has now been shown by Efthymiou, Hayes,Štefankovič and Vigoda that w.h.p. $q \approx 1.7632\ldots d$ is sufficient.

Note that $\Delta(G_{n,d/n}) \approx \log n/\log\log n$ w.h.p.

An important issue in analysing Markov chains for generating colorings is ergodicity i.e. can the chain reach all colorings.

MCMC Algorithms for coloring (random) graphs

Dyer, Flaxman, Frieze and Vigoda (2006) showed that w.h.p. $q = o(\Delta)$ suffices for the random graph $G_{n,p}, p = d/n$.

After a series of improvements by Mossel and Sly; Efthymiou it has now been shown by Efthymiou, Hayes,Štefankovič and Vigoda that w.h.p. $q \approx 1.7632\ldots d$ is sufficient.

An important issue in analysing Markov chains for generating colorings is ergodicity i.e. can the chain reach all colorings.

DFFV showed that Glauber Dynamics (where each move changes the color of a single vertex) is ergodic w.h.p. for $q \geq d + 1$.

MCMC Algorithms for coloring (random) graphs

Dyer, Flaxman, Frieze and Vigoda (2006) showed that w.h.p. $q = o(\Delta)$ suffices for the random graph $G_{n,p}, p = d/n$.

After a series of improvements by Mossel and Sly; Efthymiou it has now been shown by Efthymiou, Hayes,Štefankovič and Vigoda that w.h.p. $q \approx 1.7632\ldots d$ is sufficient.

An important issue in analysing Markov chains for generating colorings is ergodicity i.e. can the chain reach all colorings.

DFFV showed that Glauber Dynamics (where each move changes the color of a single vertex) is ergodic w.h.p. for $q \geq d + 1$.

We now discuss improvements on this ergodicity bound..

Ergodicity of Glauber Dynamics

For a (hyper)graph H and a positive integer q we let $\Omega_q(H)$ denote the set of proper q-colorings of H.

Ergodicity of Glauber Dynamics

For a (hyper)graph H and a positive integer q we let $\Omega_q(H)$ denote the set of proper q-colorings of H.

We then let $\Gamma_q(H)$ denote the graph with vertex set $\Omega_q(H)$ and an edge ϕ, ψ whenever ϕ, ψ disagree on the color of exactly one vertex.

Ergodicity of Glauber Dynamics

For a (hyper)graph H and a positive integer q we let $\Omega_q(H)$ denote the set of proper q-colorings of H.

We then let $\Gamma_q(H)$ denote the graph with vertex set $\Omega_q(H)$ and an edge ϕ, ψ whenever ϕ, ψ disagree on the color of exactly one vertex.

Molloy (2016) proved, for the case $k = 2$,

$$\text{if } q \lesssim \frac{d}{\log d} \text{ then } \Gamma_q(G_{n,d/n}) \text{ has no giant component.}$$

Here the asymptotics $\lesssim$ are with respect to growing d.

Ergodicity of Glauber Dynamics

For a (hyper)graph H and a positive integer q we let $\Omega_q(H)$ denote the set of proper q-colorings of H.

We then let $\Gamma_q(H)$ denote the graph with vertex set $\Omega_q(H)$ and an edge ϕ, ψ whenever ϕ, ψ disagree on the color of exactly one vertex.

Molloy (2016) proved, for the case $k = 2$,

$$\text{if } q \lesssim \frac{d}{\log d} \text{ then } \Gamma_q(G_{n,d/n}) \text{ has no giant component.}$$

Here the asymptotics $\lesssim$ are with respect to growing d.

Recall that $d/\log d$ is the approximate number of colors required by the Greedy Coloring Algorithm.

Ergodicity of Glauber Dynamics

Let

$$\alpha = \left(\frac{(k-1)d}{\log d - 5(k-1)\log\log d} \right)^{\frac{1}{k-1}}, \quad \beta = 3\log^{3k} d.$$

Theorem (Anastos and Frieze (2018))

If $k \geq 2$ and $p = \frac{d}{\binom{n-1}{k-1}}$ and $d = O(1)$ is sufficiently large, then

(i) *If $q \geq \alpha + \beta + 1$ then w.h.p. $\Gamma_q(H_{n,p;k})$ is connected.*

(ii) *If $q \geq \alpha + 2\beta + 1$ then the diameter of $\Gamma_q(H_{n,p;k})$ is $O(n)$ w.h.p.*

Ergodicity of Glauber Dynamics

α, β-colorability: Let $V_1, V_2, \ldots, V_\alpha$ be a sequence of independent sets of H such that for each $j \geq 1$, V_j is a maximal independent subset of $H - V_{<j}$. We call this a maximally independent sequence of length α.

Ergodicity of Glauber Dynamics

α, β-colorability: Let $V_1, V_2, \ldots, V_\alpha$ be a sequence of independent sets of H such that for each $j \geq 1$, V_j is a maximal independent subset of $H - V_{<j}$. We call this a maximally independent sequence of length α.

The first α color classes chosen by the greedy algorithm will be a maximally independent sequence of length α.

α, β-colorability: Let $V_1, V_2, \ldots, V_\alpha$ be a sequence of independent sets of H such that for each $j \geq 1$, V_j is a maximal independent subset of $H - V_{<j}$. We call this a maximally independent sequence of length α.

The first α color classes chosen by the greedy algorithm will be a maximally independent sequence of length α.

We say that a hypergraph H is (α, β)-colorable if **there does not exist** a maximally independent sequence of length α such that $H - V_{\leq\alpha}$ contains a β-core.
(A β-core is set of vertices that induces a hypergraph of minimum degree $\geq \beta$.)

Ergodicity of Glauber Dynamics

Theorem

If H is (α, β)-colorable and $q \geq \alpha + \beta + 1$ then $\Gamma(H, q)$ is connected.

Ergodicity of Glauber Dynamics

Theorem

If H is (α, β)-colorable and $q \geq \alpha + \beta + 1$ then $\Gamma(H, q)$ is connected.

Theorem

W.h.p. $H_{n,d/n;k}$ is (α, β)-colorable for the given values of α, β.

Proof via a few first moment calculations.

$$\alpha = \left(\frac{(k-1)d}{\log d - 5(k-1) \log \log d} \right)^{\frac{1}{k-1}}, \quad \beta = 3 \log^{3k} d.$$

Ergodicity of Glauber Dynamics

Definition

A coloring with color sets $V_1, V_2, \dots, V_{\alpha+\beta}$ is said to be a *good greedy coloring* if (i) $V_1, V_2, \dots, V_\alpha$ is a maximally independent sequence of length α and (ii) $V \setminus \bigcup_{\ell \le \alpha} V_\ell$ has no β-core.

Ergodicity of Glauber Dynamics

Definition

A coloring with color sets $V_1, V_2, \ldots, V_{\alpha+\beta}$ is said to be a *good greedy coloring* if (i) $V_1, V_2, \ldots, V_\alpha$ is a maximally independent sequence of length α and (ii) $V \setminus \bigcup_{\ell \leq \alpha} V_\ell$ has no β-core.

Theorem

Let H be an (α, β)-colorable hypergraph, $q \geq \alpha + \beta + 1$ and χ be a $[q]$-coloring of H. Then there exists a good greedy coloring τ of H such that there exists a path in $\Gamma_q(H)$ from χ to τ.

Theorem

Let H be an (α, β)-colorable hypergraph, $q \geq \alpha + \beta + 1$ and let χ, τ be two good greedy colorings. Then there exists a path from χ to τ in $\Gamma_q(H)$.

Ergodicity of Glauber Dynamics

Theorem

Let H be an (α, β)-colorable hypergraph, $q \geq \alpha + \beta + 1$ and χ be a $[q]$-coloring of H. Then there exists a good greedy coloring τ of H such that there exists a path in $\Gamma_q(H)$ from χ to τ.

Ergodicity of Glauber Dynamics

Theorem

Let H be an (α, β)-colorable hypergraph, $q \geq \alpha + \beta + 1$ and χ be a $[q]$-coloring of H. Then there exists a good greedy coloring τ of H such that there exists a path in $\Gamma_q(H)$ from χ to τ.

Let $C_1, \ldots, C_q$ be the color classes of χ. Let $V_1 \supseteq C_1$ be a maximal independent set containing C_1. Re-color $V_1 \setminus C_1$ with color 1. Then $C_i \leftarrow C_i \setminus V_1, i \geq 2$.

Theorem

Let H be an (α,β)-colorable hypergraph, $q \geq \alpha+\beta+1$ and χ be a $[q]$-coloring of H. Then there exists a good greedy coloring τ of H such that there exists a path in $\Gamma_q(H)$ from χ to τ.

Let $C_1,\ldots,C_q$ be the color classes of χ. Let $V_1 \supseteq C_1$ be a maximal independent set containing C_1. Re-color $V_1 \setminus C_1$ with color 1. Then $C_i \leftarrow C_i \setminus V_1, i \geq 2$.

Repeat another $\alpha - 1$ times to create χ'. This will follow a path in Γ.

Ergodicity of Glauber Dynamics

Theorem

Let H be an (α, β)-colorable hypergraph, $q \geq \alpha + \beta + 1$ and χ be a $[q]$-coloring of H. Then there exists a good greedy coloring τ of H such that there exists a path in $\Gamma_q(H)$ from χ to τ.

Let $C_1, \ldots, C_q$ be the color classes of χ. Let $V_1 \supseteq C_1$ be a maximal independent set containing C_1. Re-color $V_1 \setminus C_1$ with color 1. Then $C_i \leftarrow C_i \setminus V_1, i \geq 2$.

Repeat another $\alpha - 1$ times to create χ'. This will follow a path in Γ.

$W = V \setminus \bigcup_{i=1}^{\alpha} V_i$ has no β-core and we can re-color it from $[\alpha + 1, \alpha + \beta]$ to give us a good greedy coloring. This needs a liitle explanation.

Ergodicity of Glauber Dynamics

Let $W = V \setminus \bigcup_{1\leq i\leq \alpha} V_i$. Because H is (α, β)-colorable, we find that W has no β-core. Because W has no β-core there exists a proper coloring τ' of the subgraph of H induced by W that uses only colors in $[\alpha + \beta] \setminus [\alpha]$. Set τ to be the coloring that agrees with χ' on $V \setminus W$ and with τ' on W.

Ergodicity of Glauber Dynamics

Let $W = V \setminus \bigcup_{1 \leq i \leq \alpha} V_i$. Because H is (α, β)-colorable, we find that W has no β-core. Because W has no β-core there exists a proper coloring τ' of the subgraph of H induced by W that uses only colors in $[\alpha + \beta] \setminus [\alpha]$. Set τ to be the coloring that agrees with χ' on $V \setminus W$ and with τ' on W.

For $0 \leq i \leq r$ let τ_i agree with τ on $\{v_1, ..., v_i\}$ and with χ on $\{v_{i+1}, ..., v_r\}$. On $V \setminus W$ it agrees with both. $\tau_0 = \chi$ and $\tau_r = \tau$.

Ergodicity of Glauber Dynamics

Let $W = V \setminus \bigcup_{1 \le i \le \alpha} V_i$. Because H is (α, β)-colorable, we find that W has no β-core. Because W has no β-core there exists a proper coloring τ' of the subgraph of H induced by W that uses only colors in $[\alpha + \beta] \setminus [\alpha]$. Set τ to be the coloring that agrees with χ' on $V \setminus W$ and with τ' on W.

For $0 \le i \le r$ let τ_i agree with τ on $\{v_1, ..., v_i\}$ and with χ on $\{v_{i+1}, ..., v_r\}$. On $V \setminus W$ it agrees with both. $\tau_0 = \chi$ and $\tau_r = \tau$. By induction there is a sequence of changes from χ to τ_i that is proper except perhaps for $\{v_{i+1}, \ldots, v_r\}$.

Ergodicity of Glauber Dynamics

Let $W = V \setminus \bigcup_{1 \leq i \leq \alpha} V_i$. Because H is (α, β)-colorable, we find that W has no β-core. Because W has no β-core there exists a proper coloring τ' of the subgraph of H induced by W that uses only colors in $[\alpha + \beta] \setminus [\alpha]$. Set τ to be the coloring that agrees with χ' on $V \setminus W$ and with τ' on W.

For $0 \leq i \leq r$ let τ_i agree with τ on $\{v_1, ..., v_i\}$ and with χ on $\{v_{i+1}, ..., v_r\}$. On $V \setminus W$ it agrees with both. $\tau_0 = \chi$ and $\tau_r = \tau$. By induction there is a sequence of changes from χ to τ_i that is proper except perhaps for $\{v_{i+1}, \ldots, v_r\}$.

To go from i to $i + 1$, follow i sequence and re-color v_{i+1} whenever it threatens to cause an improper coloring. Give v_{i+1} its τ color at the end of the sequence.

Ergodicity of Glauber Dynamics

Lemma

Let H be an (α, β)-colorable hypergraph, $q \geq \alpha + \beta + 1$ and let χ, τ be two good greedy colorings. Then there exists a path from χ to τ in $\Gamma_q(H)$.

There exists a maximal independent sequence $V_1, V_2, \ldots, V_\alpha$ of length α such that if $V' = V \setminus \bigcup_{1 \leq i \leq \alpha} V_i$ then (i) for $i \in [a]$, τ assigns the color i to $v \in V_i$ and (ii) τ assigns only colors in $[\alpha + \beta] \setminus [\alpha]$ to vertices in V'.

Ergodicity of Glauber Dynamics

Lemma

Let H be an (α,β)-colorable hypergraph, $q \geq \alpha+\beta+1$ and let χ,τ be two good greedy colorings. Then there exists a path from χ to τ in $\Gamma_q(H)$.

There exists a maximal independent sequence $V_1, V_2, \ldots, V_\alpha$ of length α such that if $V' = V \setminus \bigcup_{1\leq i\leq\alpha} V_i$ then (i) for $i \in [a]$, τ assigns the color i to $v \in V_i$ and (ii) τ assigns only colors in $[\alpha+\beta] \setminus [\alpha]$ to vertices in V'.

Let c be a color not assigned by χ. Starting from χ we recolor all vertices that are colored 1 by color c to create $\bar{\chi}$. Then we continue from $\bar{\chi}$ by recoloring all the vertices in V_1 by color 1 and we let χ' be the resulting coloring. Clearly there is a path P_1 from χ to χ' in Γ.

Ergodicity of Glauber Dynamics

Now V_1 is colored the same in both.

Ergodicity of Glauber Dynamics

Now V_1 is colored the same in both.

Let χ_1' be the coloring of $V \setminus V_1$ induced by χ'. Transform χ_1' to a good greedy $(\alpha - 1, \beta)$ coloring and then use induction on α to reduce to the $(0, \beta)$ case.

Now V_1 is colored the same in both.

Let χ_1' be the coloring of $V \setminus V_1$ induced by χ'. Transform χ_1' to a good greedy $(\alpha - 1, \beta)$ coloring and then use induction on α to reduce to the $(0, \beta)$ case.

The $(0, \beta)$ case involves re-coloring a hypergraph without a β-core and this has been dealt with.

Randomly Coloring Arbitrary Simple Hypergraphs

We discuss using Glauber Dynamics to randomly color an arbitrary simple hypergraph.

Randomly Coloring Arbitrary Simple Hypergraphs

We discuss using Glauber Dynamics to randomly color an arbitrary simple hypergraph.

Frieze and Melsted (2011) gave examples of blocked colorings:

Theorem

Let $k \geq 3$ and let m, q be sufficiently large. Suppose that $\epsilon \leq \frac{1}{10k!}$. Then there exists a hypergraph H with qm vertices and maximum degree $\Delta \in [\frac{\epsilon qm}{2(k-1)!}, \frac{2\epsilon qm}{(k-1)!}]$ and a coloring with q colors so that there are no Glauber moves.

Randomly Coloring Arbitrary Simple Hypergraphs

We discuss using Glauber Dynamics to randomly color an arbitrary simple hypergraph.

Frieze and Melsted (2011) gave examples of blocked colorings:

Theorem

Let $k \geq 3$ and let m, q be sufficiently large. Suppose that $\epsilon \leq \frac{1}{10k!}$. Then there exists a hypergraph H with qm vertices and maximum degree $\Delta \in [\frac{\epsilon qm}{2(k-1)!}, \frac{2\epsilon qm}{(k-1)!}]$ and a coloring with q colors so that there are no Glauber moves.

So, for small q we have to be satisfied with generating a (near) random coloring from a giant component of Γ.

Randomly Coloring Arbitrary Simple Hypergraphs

Theorem

Let H be a k-uniform simple hypergraph with maximum degree Δ where $k \geq 3$. Suppose that

$$2\Delta \geq q \geq \max\left\{C_k \log n, 10k\epsilon_k^{-1}\Delta^{1/(k-1)}\right\}.$$

Suppose that the initial coloring X_0 is chosen randomly from q^V. Then for an arbitrary constant $\delta > 0$ we have

$$d_{TV}(X_t, Y) \leq \delta$$

for $t \geq t_\delta$, where $t_\delta = 2n\log(2n/\delta)$.

Randomly Coloring Arbitrary Simple Hypergraphs

Let X be a coloring of V. For a vertex $v \in V$ and $1 \leq i \leq k-1$

$$E_{v,i,X} = \{e : \ v \in e \ and \ |\{X(w) : w \in e \setminus \{v\}\}| = i\}$$

be the set of edges e containing v in which $e \setminus \{v\}$ uses exactly i distinct colors under X.

Randomly Coloring Arbitrary Simple Hypergraphs

Let X be a coloring of V. For a vertex $v \in V$ and $1 \leq i \leq k-1$

$$E_{v,i,X} = \{e : \ v \in e \; and \; |\{X(w) : w \in e \setminus \{v\}\}| = i\}$$

be the set of edges e containing v in which $e \setminus \{v\}$ uses exactly i distinct colors under X.

Let $y_{v,i,X} = |E_{v,i,X}|$, so that the number of bad colors for v is given by $|B(v,X)| = y_{v,1,X}$ for all v, X.

Randomly Coloring Arbitrary Simple Hypergraphs

Let X be a coloring of V. For a vertex $v \in V$ and $1 \leq i \leq k-1$

$$E_{v,i,X} = \{e: \ v \in e \ and \ |\{X(w) : w \in e \setminus \{v\}\}| = i\}$$

be the set of edges e containing v in which $e \setminus \{v\}$ uses exactly i distinct colors under X.

Let $y_{v,i,X} = |E_{v,i,X}|$, so that the number of bad colors for v is given by $|B(v,X)| = y_{v,1,X}$ for all v, X.

We define the sequence $\boldsymbol{\epsilon} = \epsilon_k, \epsilon_k^2, \ldots, \epsilon_k^{k-2}$.

Definition

We say that X is $\boldsymbol{\epsilon}$-*bad* if $\exists v \in V, 1 \leq i \leq k-2$ such that

$$y_{v,i,X} \geq \mu_i \text{ where } \mu_i = (\epsilon_k q)^i.$$

Otherwise we say that X is $\boldsymbol{\epsilon}$-*good*.

Randomly Coloring Arbitrary Simple Hypergraphs

We start the chain X_t with a random q-coloring from q^V and then couple it against a random proper coloring Y_t.

Randomly Coloring Arbitrary Simple Hypergraphs

We start the chain X_t with a random q-coloring from q^V and then couple it against a random proper coloring Y_t.

If we know that both X_t, Y_t are ϵ-good then we can show in a straightforward manner that their Hamming distance decreases in expectation in a single iteration.

Randomly Coloring Arbitrary Simple Hypergraphs

We start the chain X_t with a random q-coloring from q^V and then couple it against a random proper coloring Y_t.

If we know that both X_t, Y_t are ϵ-good then we can show in a straightforward manner that their Hamming distance decreases in expectation in a single iteration.

To show that a random proper coloring is ϵ-good w.h.p. we use the local lemma in the following way.

Randomly Coloring Arbitrary Simple Hypergraphs

Let $\mathbf{Pr}_\Omega$ refer to uniform probability on q^V and let $\mathbf{Pr}_\mathcal{Q}$ refer to uniform probability on proper colorings.

Randomly Coloring Arbitrary Simple Hypergraphs

Let $\mathbf{Pr}_\Omega$ refer to uniform probability on q^V and let $\mathbf{Pr}_\mathcal{Q}$ refer to uniform probability on proper colorings.

Consider a random coloring $X \in q^V$. For a vertex $v \in V$ we let $\mathcal{A}_v = \mathcal{A}_\epsilon(v)$ denote the event $\{v \text{ is not } \epsilon - good\}$. For an edge $e \in E$ we let $\mathcal{B}_e$ denote the event $\{e \text{ is not properly colored}\}$.

Randomly Coloring Arbitrary Simple Hypergraphs

Let $\mathbf{Pr}_\Omega$ refer to uniform probability on q^V and let $\mathbf{Pr}_\mathcal{Q}$ refer to uniform probability on proper colorings.

Consider a random coloring $X \in q^V$. For a vertex $v \in V$ we let $\mathcal{A}_v = \mathcal{A}_\epsilon(v)$ denote the event $\{v \text{ is not } \epsilon - good\}$. For an edge $e \in E$ we let $\mathcal{B}_e$ denote the event $\{e \text{ is not properly colored}\}$.

$$p = \mathbf{Pr}_\Omega(\mathcal{B}_e) = \frac{1}{q^{k-1}}.$$

Randomly Coloring Arbitrary Simple Hypergraphs

Let $\mathbf{Pr}_\Omega$ refer to uniform probability on q^V and let $\mathbf{Pr}_\mathcal{Q}$ refer to uniform probability on proper colorings.

Consider a random coloring $X \in q^V$. For a vertex $v \in V$ we let $\mathcal{A}_v = \mathcal{A}_\epsilon(v)$ denote the event $\{v \text{ is not } \epsilon - good\}$. For an edge $e \in E$ we let $\mathcal{B}_e$ denote the event $\{e \text{ is not properly colored}\}$.

$$p = \mathbf{Pr}_\Omega(\mathcal{B}_e) = \frac{1}{q^{k-1}}.$$

If $x_e = 2/q^{k-1}$ then

$$p \leq x_e \prod_{f \in E, f \cap e \neq \emptyset} (1 - x_f).$$

Randomly Coloring Arbitrary Simple Hypergraphs

It follows from Haeupler, Saha and Srinivasan (2011) that

$$\mathbf{Pr}_{\mathcal{Q}}(\mathcal{A}_v) \leq \mathbf{Pr}_{\Omega}(\mathcal{A}_v) \prod_{f \in \mathcal{N}_v} (1 - x_f)^{-1},$$

where $\mathcal{N}_v = \{f : f \cap e \neq \emptyset \text{ and } f \cap e \text{ for some } e \ni v\}$.

The proof of HSS is a straightforward adaptation of the usual Local Lemma proof.

It follows from Haeupler, Saha and Srinivasan (2011) that

$$\mathbf{Pr}_{\mathcal{Q}}(\mathcal{A}_v) \leq \mathbf{Pr}_{\Omega}(\mathcal{A}_v) \prod_{f \in \mathcal{N}_v} (1 - x_f)^{-1},$$

where $\mathcal{N}_v = \{f : f \cap e \neq \emptyset \text{ and } f \cap e \text{ for some } e \ni v\}$.

The proof of HSS is a straightforward adaptation of the usual Local Lemma proof.

For our given q, $\mathbf{Pr}_{\Omega}(\mathcal{A}_v)$ is small and then so is $\mathbf{Pr}_{\mathcal{Q}}(\mathcal{A}_v)$.

Open Questions

1. Does Glauber Dynamics succeed in polynomial time on random Hypergraphs with $q = O(n^{1/(k-1)})$ colors?

Open Questions

1. Does Glauber Dynamics succeed in polynomial time on random Hypergraphs with $q = O(n^{1/(k-1)})$ colors?
2. Remove the $\Omega(\log n)$ requirement for coloring arbitrary simple hypergraphs.

Guo, Liao, Lu and Zhang (2018) deals with deterministically, approximately, counting colorings. The requirements are

$$k \geq 28, q \geq 315\Delta^{14/(k-14)}.$$

THANK YOU