Classification for Counting Problems

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Thanks to many collaborators: Xi Chen, Zhiguo Fu, Kurt Girstmair, Heng Guo, Mike Kowalczyk, Pinyan Lu, Tyson Williams . . .

Martin Dyer's Pioneering Wrok

Martin Dyer made foundational contributions to the classification program of counting problems.
His paper with Catherine Greenhill

- The complexity of counting graph homomorphisms. (2000).

marked the beginning.

Martin's Amazingly Broad Contribution

Martin's contributions span an amazingly wide range of topics within TCS.

His work opened new frontiers, established landmark results of great originality and beauty that they have become prominent textbook material.

Repeatedly, his work displayed deep insight that the full impact of which magnifies as the years and decades progress.

They introduced research directions and themes of lasting influence.

Time and again, Martin's work changed the research landscape.

Classification Program for Counting problems

The aim of the Classification Program is to classify every problem in a broad complexity class.

This will include most counting problems at the NP level, expressible as a sum-of-product computation, such as Independent Sets, k-SAT, k-vertex-colorings, k-edge-colorings, vertex covers, matchings, perfect matchings, cycle covers, partition functions from statistical mechanics ..., graph homomorphisms, constraint satisfaction problems, ...,

Complexity Classification Theorem

Theorem (Dyer-Greenhill)

For every square 0-1 symmetric matrix H, the number of graph homomorphisms from G to H

$$Z_H(G) = \sum_{\sigma: V(G) \to V(H)} \prod_{(u,v) \in E(G)} H(\sigma(u), \sigma(v))$$

is either in P-time or #P-complete.

A Number Theory Problem

Let $0 < \varphi < \psi < \pi/2$ denote two angles. Then

$$0 < \tan(\varphi) < \tan(\psi) < \infty.$$

Question:

Is it possible that

$$\tan(\psi) = 2\tan(\varphi),$$

and yet φ and ψ are both rational multiples of π?

We prove that φ and ψ *cannot* be both rational multiples of π.

Is this Obvious?

Such concrete results are used to prove our complexity classification theorems.

Is it Obvious? ... in the eye of the beholder

$$\sin\frac{\pi}{60} = \frac{(2-\sqrt{12})\sqrt{5+\sqrt{5}}+(\sqrt{10}-\sqrt{2})(\sqrt{3}+1)}{16}$$

$$\sin\frac{\pi}{30} = \frac{\sqrt{30-\sqrt{180}}-\sqrt{5}-1}{8}, \quad \sin\frac{\pi}{20} = \frac{\sqrt{10}+\sqrt{2}-\sqrt{20-\sqrt{80}}}{8}$$

$$\sin\frac{\pi}{15} = \frac{\sqrt{10+\sqrt{20}}+\sqrt{3}-\sqrt{15}}{8}, \quad \sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4} = \tfrac{1}{2}\varphi^{-1}$$

$$\sin\frac{7\pi}{60} = \frac{(2+\sqrt{12})\sqrt{5-\sqrt{5}}-(\sqrt{10}+\sqrt{2})(\sqrt{3}-1)}{16}, \quad \sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

$$\sin\frac{2\pi}{15} = \frac{\sqrt{3}+\sqrt{15}-\sqrt{10-\sqrt{20}}}{8}, \quad \sin\frac{3\pi}{20} = \frac{\sqrt{20+\sqrt{80}}-\sqrt{10}+\sqrt{2}}{8}$$

$$\sin\frac{11\pi}{60} = \frac{(\sqrt{12}-2)\sqrt{5+\sqrt{5}}+(\sqrt{10}-\sqrt{2})(\sqrt{3}+1)}{16}, \quad \sin\frac{\pi}{5} = \frac{\sqrt{10-\sqrt{20}}}{4}$$

$$\sin\frac{13\pi}{60} = \frac{(2-\sqrt{12})\sqrt{5-\sqrt{5}}+(\sqrt{10}+\sqrt{2})(\sqrt{3}+1)}{16}, \quad \sin\frac{7\pi}{30} = \frac{\sqrt{30+\sqrt{180}}-\sqrt{5}+1}{8}$$

$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \cos\frac{\pi}{4}$$

$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta = \sqrt{1-\sin^2\theta}$$

Spin Systems on Graphs

Given a graph $G = (V, E)$.

Suppose there is a binary constraint function $f : \{0,1\}^2 \to \mathbb{R}$ assigned to each edge.

Consider all vertex assignments $\sigma : V \to \{0,1\}$.

For each $(u, v) \in E$, an assignment σ gives an evaluation

$$\prod_{(u,v)\in E} f(\sigma(u), \sigma(v)).$$

Then the partition function of the Spin System is

$$Z_f(G) = \sum_{\sigma:V\to\{0,1\}} \prod_{(u,v)\in E(G)} f(\sigma(u), \sigma(v)).$$

Some Examples of Spin Systems

A binary constraint function f can be represented by a matrix

$$M(f) = \begin{bmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{bmatrix}.$$

For example:

If $M(f) = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, then the value of the partition function is the number of independent sets of G;

If $M(f) = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, then the value of the partition function is the number of anti-chains of a partially ordered set.

If $M(f) = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$, then the problem is essentially the number of even indeced subgraphs.

A Classification Theorem

Theorem

Let $M(f) = \left[\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right]$, where $w, x, y, z \in \mathbb{R}$. Then $Z_f(G)$ on k-regular graphs, for any $k \geq 3$, is either

- $\#$P*-hard, or*
- *P-time computable:*
  1. *f is of product type, $f \in \mathscr{P}$: $wz = xy$, or $w = z = 0$, or $x = y = 0$;*
  2. *f is of affine type, $f \in \mathscr{A}$: $w^2 = x^2 = y^2 = z^2$.*

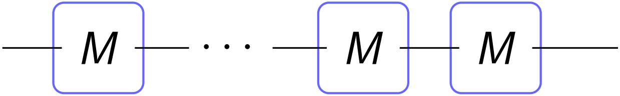
Restricted to planar graphs, then additionally it is P-time computable if

- *It is transformable to matchgates, $\mathscr{M}$-transformable:*
  $w = \epsilon z, x = \epsilon y$, or k is even and $w = \epsilon z, x = -\epsilon y$, where $\epsilon = \pm 1$.

But everything else remains $\#$P*-hard.*

Interpolation and the Ratio of Eigenvalues

Given $M(f) = \begin{bmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{bmatrix}$. Using f one can construct gadgets with a signature matrix $M = \begin{bmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{bmatrix}$.



To prove #P-hardness, we need to ensure that there is no $n \in \mathbb{N}$ such that

$$M^n = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where λ is a constant.

For nonsingular matrices, this is equivalent to:

The ratio of the two eigenvalues of M is not a root of unity.

Two Constructions

We can construct two gadgets with matrices respectively:

$$\begin{aligned} M(g_1) &= \begin{bmatrix} 1-x^2 & 2x \\ -2x & 1-x^2 \end{bmatrix}, \\ M(g_2) &= \begin{bmatrix} 1-x^4 & x+x^3 \\ -x-x^3 & 1-x^4 \end{bmatrix} \\ &= \lambda(x) \begin{bmatrix} 1-x^2 & x \\ -x & 1-x^2 \end{bmatrix}, \end{aligned}$$

where $x \in \mathbb{R}$, and $\lambda(x) = 1 + x^2$.

Let

$$a = 1 - x^2, \qquad \text{and} \qquad b = x.$$

The ratios of the eigenvalues of $M(g_1)$ and $M(g_2)$ are

$$\rho_1 = \frac{\lambda_1}{\mu_1} = \frac{a + b\mathrm{i}}{a - b\mathrm{i}} = e^{\mathrm{i}2\varphi} \qquad \text{and} \qquad \rho_2 = \frac{\lambda_2}{\mu_2} = \frac{a + 2b\mathrm{i}}{a - 2b\mathrm{i}} = e^{\mathrm{i}2\psi}.$$

Then

$$\cot(\varphi) = \frac{a}{b} \qquad \text{and} \qquad \cot(\psi) = \frac{a}{2b}.$$

ρ_1 and ρ_2 are roots of unity $\iff$ φ and ψ are rational multiples of π.

We will prove that it is impossible that both ρ_1 and ρ_2 are roots of unity.

Theorem

Suppose $0 < \varphi < \psi < \pi/2$, *and* $\cot(\varphi) = r\cot(\psi)$, *for some* $r \in \mathbb{Q}$ *and* $r \neq 3$. *Then* φ *and* ψ *are not both rational multiples of* π.

The exception is real:

$$\cot\frac{\pi}{6} = \sqrt{3} = \frac{3}{\sqrt{3}} = 3\cot\frac{\pi}{3}.$$

Carl L. Siegel

Siegel proved (1949): The values $\cot(k\pi/n)$
(for $1 \leq k < n/2$, $\gcd(k, n) = 1$) are $\mathbb{Q}$-linearly independent.

Chowla extended (1964) (1970) these results.

Hasse (1971) proved similar theorems for tangent values $\tan(k\pi/p)$, any fixed prime p.

Jager and Lenstra (1975) proved similar theorems for cosecant values $\csc(2k\pi/p)$ (for $1 \leq k \leq (p-1)/2$).

Girstmair (1987) gave a representation theoretic treatment to the problem.

But these theorems do not suffice for what we need.

Our problem expressed in cyclotomic fields

Suppose $\varphi = \frac{k\pi}{n}$ and $\psi = \frac{k'\pi}{n'}$, where $\gcd(k, n) = 1, \gcd(k', n') = 1$,

The question is: Can $\cot(\varphi)$ be a rational multiple of $\cot(\psi)$?

Let $\zeta_n = e^{2\pi i/n}$. Let $\Phi_n = \mathbb{Q}(\zeta_n)$ be the n-th cyclotomic field.

Since

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}, \qquad \cos(x) = \frac{e^{ix} + e^{-ix}}{2},$$

It follows easily that

$$i \cot\left(\frac{k\pi}{n}\right) = \frac{1 + \zeta_n^k}{1 - \zeta_n^k} \in \Phi_n.$$

$\mathfrak{z} \mapsto \frac{1+\mathfrak{z}}{1-\mathfrak{z}}$ is a Möbius transformation.

Let $t = \mathrm{i}\cot(\varphi) = \mathrm{i}\cot(\frac{k\pi}{n}) \in \Phi_n$, and $t' = \mathrm{i}\cot(\psi) = \mathrm{i}\cot(\frac{k'\pi}{n'}) \in \Phi_{n'}$.

Then by

$$t = \frac{1+\zeta_n^k}{1-\zeta_n^k}, \qquad t' = \frac{1+\zeta_{n'}^{k'}}{1-\zeta_{n'}^{k'}},$$

$$\zeta_n^k = \frac{t-1}{t+1}, \qquad \zeta_{n'}^{k'} = \frac{t'-1}{t'+1},$$

Suppose $\cot(\varphi) = r\cot(\psi)$, for some $r \in \mathbb{Q}$. Then

$$\mathbb{Q}(\zeta_n) = \mathbb{Q}(\zeta_n^k) = \mathbb{Q}(t) = \mathbb{Q}(t') = \mathbb{Q}(\zeta_{n'}^{k'}) = \mathbb{Q}(\zeta_{n'}).$$

Equality of two cyclotomic fields

We have

$$\mathbb{Q}(\zeta_n) = \mathbb{Q}(\zeta_{n'}).$$

Theorem

If $\mathbb{Q}(\zeta_n) = \mathbb{Q}(\zeta_{n'})$, then either $n = n'$, or n is odd and $n' = 2n$, or n' is odd and $n = 2n'$.

I will skip the case $n = n'$, and only discuss the case $n' = 2n$ for odd n.

A Dirichlet character to the modulus m is any function χ from $\mathbb{Z}$ to $\mathbb{C}$ such that χ has the following properties:

- $\chi(k) = \chi(k+m)$ for all $k \in \mathbb{Z}$.
- If $\gcd(k,m) > 1$ then $\chi(k) = 0$; if $\gcd(m,k) = 1$ then $\chi(k) \neq 0$.
- $\chi(k\ell) = \chi(k)\chi(\ell)$ for all integers k and ℓ.

A Dirichlet character χ is said to be *odd* if $\chi(-1) = -1$.

Dirichlet characters are used to define the Dirichlet L-series

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

For any odd Dirichlet character χ to the modulus n,
Let $t \in \Phi_n$, the Leopoldt's character coordinates $y(\chi \mid t) \in \mathbb{C}$ are defined by

$$y(\chi \mid t)\, \mathfrak{g}(\overline{\chi_d}) = \sum_{1 \le j \le n \ \gcd(j,n)=1} \overline{\chi(j)} \sigma_j(t),$$

where d is the *conductor* of χ, χ_d is the induced primitive character of χ mod d, overline denotes complex conjugation, the value

$$\mathfrak{g}(\overline{\chi_d}) = \sum_{j=1}^{d} \overline{\chi_d}(j) e^{-2\pi i j/d} \neq 0$$

is the Gauss sum, and σ_j is the automorphism in the Galois group $\mathrm{Gal}(\Phi_n/\mathbb{Q})$ that maps ζ_n to ζ_n^j.

Note that for $r \in \mathbb{Q}$, $y(\chi \mid rt) = r y(\chi \mid t)$.

Under a group action σ_k

For $\gcd(k,n)=1$, we have an automorphism $\sigma_k : \zeta_n \mapsto \zeta_n^k$ in the Galois group $\mathrm{Gal}(\Phi_n/\mathbb{Q})$.

If $t_k = \mathrm{i}\cot(\frac{k\pi}{n}) = \frac{1+\zeta_n^k}{1-\zeta_n^k} \in \Phi_n$, and $t_1 = \mathrm{i}\cot(\frac{\pi}{n}) = \frac{1+\zeta_n}{1-\zeta_n}$, then $t_k = \sigma_k(t_1)$.

For a fixed $k \in \mathbb{Z}_n^\times$, $\sigma_j \circ \sigma_k = \sigma_{jk}$ runs through all $\mathrm{Gal}(\Phi_n/\mathbb{Q})$, when j runs through $\mathbb{Z}_n^\times$. Then

$$\begin{aligned}
\sum_{j\in\mathbb{Z}_n^\times} \overline{\chi(j)}\sigma_j(t_k) &= \chi(k)\sum_{j\in\mathbb{Z}_n^\times} \chi(k)^{-1}\overline{\chi(j)}\sigma_j(\sigma_k(t_1)) \\
&= \chi(k)\sum_{j\in\mathbb{Z}_n^\times} \overline{\chi(kj)}\sigma_{jk}(t_1) \\
&= \chi(k)\sum_{j\in\mathbb{Z}_n^\times} \overline{\chi(j)}\sigma_j(t_1).
\end{aligned}$$

Since the Gauss sum $\mathfrak{g}(\overline{\chi_d}) \neq 0$,

$$y(\chi \mid t_k) = \chi(k)y(\chi \mid t_1).$$

Odd Dirichlet characters

Dirichlet characters mod n form a group, isomorphic to $\mathbb{Z}_n^\times$. The character groups of $\mathbb{Z}_n^\times$ and $\mathbb{Z}_{2n}^\times$ are isomorphic, for odd n.

By the group structure, it is known that an odd Dirichlet character χ on $\mathbb{Z}_n^\times$ exists.

Since n is an induced modulus, and odd, the conductor d of χ is also odd.

Take any odd Dirichlet character χ mod $2n$. Girstmair proved that

$$y(\chi \mid \mathrm{i}\cot(\frac{\pi}{2n})) = \frac{4n}{d} \prod_{p \mid 2n} \left(1 - \frac{\overline{\chi_d(p)}}{p}\right) B_{\chi_d},$$

and

$$y(\chi \mid \mathrm{i}\cot(\frac{\pi}{n})) = \frac{2n}{d} \prod_{p \mid n} \left(1 - \frac{\overline{\chi_d(p)}}{p}\right) B_{\chi_d}.$$

Here $B_{\chi_d} = \sum_{j=1}^{d} \chi_d(j) j/d$ is the generalized Bernoulli number.

It is known that $B_{\chi_d} \neq 0$. (Equivalent to $L(1, \chi_d) \neq 0$.)

$$y(\chi \mid \mathrm{i}\cot(\frac{k'\pi}{2n}) = \chi(k')y(\chi \mid \mathrm{i}\cot(\frac{\pi}{2n})) = \chi(k')\frac{4n}{d}\prod_{p|2n}\left(1 - \frac{\overline{\chi_d(p)}}{p}\right) B_{\chi_d},$$

$$y(\chi \mid \mathrm{i}\cot(\frac{k\pi}{n})) = \chi(k)y(\chi \mid \mathrm{i}\cot(\frac{\pi}{n})) = \chi(k)\frac{2n}{d}\prod_{p|n}\left(1 - \frac{\overline{\chi_d(p)}}{p}\right) B_{\chi_d}.$$

From $B_{\chi_d} \neq 0$, it follows that

$$\frac{y(\chi \mid \mathrm{i}\cot(\frac{k'\pi}{2n}))}{y(\chi \mid \mathrm{i}\cot(\frac{k\pi}{n}))} = \frac{\chi(k')}{\chi(k)}2\left(1 - \frac{\overline{\chi_d(2)}}{2}\right).$$

So, taking the norm squared, we get $|2 - \overline{\chi_d(2)}|^2$.

On the other hand, by assumption

$$\cot(\frac{k\pi}{n}) = \frac{a}{b}\cot(\frac{k'\pi}{2n})$$

for integers a and b. So we have

$$y(\chi \mid \mathfrak{i}\cot(\frac{k\pi}{n})) = \frac{a}{b}\, y(\chi \mid \mathfrak{i}\cot(\frac{k'\pi}{2n})).$$

Hence,

$$b^2 = a^2 \cdot |2 - \overline{\chi_d(2)}|^2. \quad (1)$$

Since χ_d is primitive mod d, and d is odd, we have $\rho = \chi_d(2) \neq 0$, which is a root of unity. We have

$$b^2 = a^2[5 - 2(\rho + \overline{\rho})].$$

Incommensurability

- If $\rho = 1$ then $a = b$, this is a contradiction to $\varphi \neq \psi$.
- If $\rho = -1$ then $b^2 = 9a^2$ or $a^2 = 9b^2$. This gives us the unique exceptional case $\varphi = \pi/6$ and $\psi = \pi/3$.
- If $\rho \neq \pm 1$, we can derive a contradiction.

Graph Homomorphism

L. Lovász:
Operations with structures, Acta Math. Hung. 18 (1967), 321-328.

http://www.cs.elte.hu/~lovasz/hom-paper.html

Let $\mathbf{A} = (A_{i,j}) \in \mathbb{C}^{\kappa \times \kappa}$ be a symmetric complex matrix.

The graph homomorphism problem is:
INPUT: An undirected graph $G = (V, E)$.
OUTPUT:

$$Z_{\mathbf{A}}(G) = \sum_{\xi : V \to [\kappa]} \prod_{(u,v) \in E} A_{\xi(u),\xi(v)}.$$

Examples of Graph Homomorphism

Let

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

then $Z_{\mathbf{A}}(G)$ counts the number of vertex covers in G.

Let

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}$$

then $Z_{\mathbf{A}}(G)$ counts the number of vertex κ-COLORINGS in G.

Dichotomy Theorem for Graph Homomorphism

Following the pionneering work of Dyer and Greenhill, a long sequence of work followed by Bulatov, Dalmau, Grohe, Goldberg, Jerrum, Thurley . . .

Theorem (C., Xi Chen and Pinyan Lu)

There is a complexity dichotomy for $Z_{\mathbf{A}}(\cdot)$:
For any symmetric complex valued matrix $\mathbf{A} \in \mathbb{C}^{\kappa\times\kappa}$, the problem of computing $Z_{\mathbf{A}}(G)$, for any input G, is either in P or #P-hard.
Given $\mathbf{A}$, whether $Z_{\mathbf{A}}(\cdot)$ is in P or #P-hard can be decided in polynomial time in the size of $\mathbf{A}$.

SIAM J. Comput. 42(3): 924-1029 (2013)

Counting CSP on general domains

For Counting CSP, Bulatov proved a dichotomy for all finite set of 0-1 constraint functions.

Dyer and Richerby gave an alternative proof for this theorem, and further proved that their dichotomy criterion is decidable.

Further generalized to all complex-valued constraint functions.

Theorem (C., Xi Chen)

Every finite set $\mathcal{F}$ of complex valued constraint functions on any finite domain set $[\kappa]$ defines a counting CSP problem $\#CSP(\mathcal{F})$ that is either computable in P or #P-hard.

J. ACM 64(3): 19:1-19:39 (2017)

The decision criteria is not known to be decidable.

It is decidable for nonnegative valued constraint functions.

#CSP on Boolean Variables

Theorem (C., Zhiguo Fu)

For any set of complex valued constraint functions $\mathcal{F}$ over Boolean variables, $\#$CSP($\mathcal{F}$) belongs to exactly one of three categories according to $\mathcal{F}$:

1. *It is P-time solvable;*
2. *It is P-time solvable over planar graphs but $\#$P-hard over general graphs;*
3. *It is $\#$P-hard over planar graphs.*

Moreover, category (2) consists precisely of those problems that are holographically reducible to the Fisher-Kasteleyn-Temperley algorithm.

STOC 2017: 842-855.

`https://arxiv.org/pdf/1603.07046.pdf` (94 pages).

Some Unexpected P-time Tractable Problems

Problem: Pl-CrazyPell

Input : A planar #CSP instance given as a bipartite graph, with a single constraint function f on 4 variables.

$$M(f) = \begin{bmatrix} 669669112435114949 & -598015350142588607 & 598015350142588611 & -669669112435114945 \\ 533639108484318913 & -476540387460305855 & 476540387460305851 & -533639108484318909 \\ -533639108484318909 & 476540387460305851 & -476540387460305855 & 533639108484318913 \\ -669669112435114945 & 598015350142588611 & -598015350142588607 & 669669112435114949 \end{bmatrix}.$$

Output : $\displaystyle\sum_{\sigma: X \to \{0,1\}} \prod_{f} f(\sigma|_X).$

Why? And How?

Let $\widehat{f} = H_2^{\otimes 4} f$, where $H_2 = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$. Then $\widehat{f}$ has the signature matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 643762416582696 & 363876031712832 & 0 \\ 0 & 569465989630582080 & 321881208291348490 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

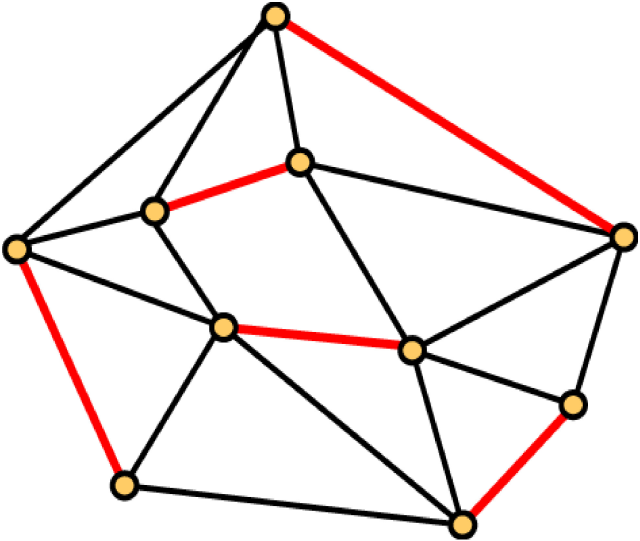
The formal reason:

Verify that $\widehat{f}$ is realizable as a matchgate signature (by Matchgate Identities). Thus #CSP(f) is tractable, by the Dichotomy Theorem.

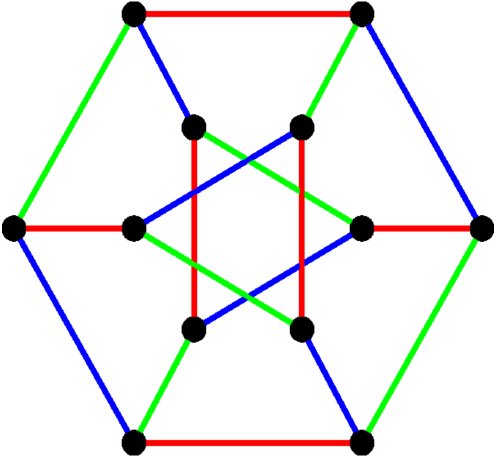
The real underlying reason:

$(32188120829134849, 1819380158564160)$ is the smallest integer solution to the Pell's equation $x^2 - 313y^2 = 1$. This enables a suitable matchgate to be constructed. And there are infinitely many such problems.

Perfect Matching



Edge Coloring



Planar Holant on Boolean Variables

Theorem (C., Zhiguo Fu, Heng Guo, Tyson Williams)

Let $\mathcal{F}$ be a set of complex-valued, symmetric functions on Boolean variables. Then there is an effective classification for all possible $\mathcal{F}$, according to which, Holant($\mathcal{F}$) *is either*

1. *P-time computable over general graphs, or*
2. *P-time computable over planar graphs but #P-hard over general graphs, or*
3. *#P-hard over planar graphs.*

However, there are two primitives for category (2). In particular, holographic reductions to FKT is NOT universal.

FOCS 2015: 1259-1276

`https://arxiv.org/abs/1505.02993` (128 pages).

Some Concrete Cases

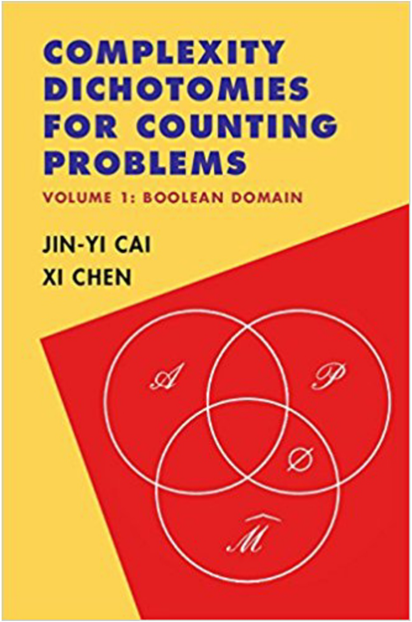
Theorem

The problem of counting perfect matchings over planar k-uniform hypergraphs is:

1. *P-time computable for $k = 2$ (ordinary graph PM).*
2. *$\#$P-complete for $k = 3, 4$.*
3. *P-time computable for all $k \geq 5$.*

More generally, if S is a set of integers specifying the hyperedge sizes, let $t = \gcd(S)$. Then counting perfect matchings is P-time computable if $t \geq 5$ or $S = \{1\}$ or $\{2\}$, and $\#$P-complete if $t \leq 4$, $S \neq \{1\}$ and $S \neq \{2\}$.

Furthermore the category $k \geq 5$ cannot be reduced to FKT.


COMPLEXITY
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XI CHEN

Happy Birthday, Martin!