B.Sc. Examination by course units

MTH5118 Probability II

Duration: 2 hours

Date and time:

You should attempt all questions on this paper

Electronic calculators may be used. The make and model should be specified on the script. The calculator must not be preprogrammed (other than by the manufacturer) prior to the examination.

You should not start reading this paper until instructed to do so by the invigilator.

You must not remove the question paper from the examination room.

You should attempt all questions. Marks awarded are shown next to the question.

1. Let X be a non-negative integer-valued random variable which has probability generating function $G_X(t) = \frac{1}{3}(2+t^2)e^{t-1}$

   [4] (a) Find $E(X)$ and $Var(X)$.

   [3] (b) Find $P(X=0)$, $P(X=1)$ and $P(X=2)$.

2. [5] A lift moves at random up and down between floors in a building with 10 floors, including the ground floor, numbered $0,1,\ldots,9$. Except when the lift reaches either the top or ground floor, each time it reaches a floor it is equally likely to go either up or down one floor.

   Gonzo enters the lift on the third floor. Find the probability that he reaches the top floor (the ninth floor) before he reaches the ground floor (floor zero).

3. [7] State the law of total probability for expectations.

   Kermit plays a series of independent games. At the start of each game he pays £1 then rolls a fair 6-sided die. If he obtains a 6 he receives £k; otherwise he receives nothing. The games continue until he throws a 1, when the series of games stop. Find the expected amount he wins and hence state the value of k for which the game is fair (the expected amount he wins is zero).

4. Each male in a certain society has at most two sons and is equally likely to have 0, 1 or 2 sons, independently of all other males. Consider the male line of descent of a specific male.

   [4] (a) Find the probability that this male has no grandsons through the male line of descent.

   [4] (b) Find the probability that his male line of descent dies out eventually.

5. X and Y are independent chi-squared random variables with parameters n and m respectively and with moment generating functions $M_X(t) = (1-2t)^{-n/2}$ and $M_Y(t) = (1-2t)^{-m/2}$. Let $Z = X+Y$.

   [3] (a) Find $M_Z(t)$ and state the distribution of Z (including any parameters).

   [3] (b) Use the moment generating function to find $E[Z]$.

6. Random variables X and Y are jointly continuous with probability density function

$$f_{X,Y}(x,y) = \begin{cases} C(1+y^2) & \text{if } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

   [3] (a) Determine if X and Y are independent.

   [6] (b) Find C, the marginal density functions for X and Y and $E[XY]$.

7. Let X_1 and X_2 be random variables with $E[X_1] = 1$, $E[X_2] = 2$, $Var(X_1) = 1$, $Var(X_2) = 4$ and $Cov(X_1, X_2) = 1$. Define $Y_1 = X_1 + X_2$ and $Y_2 = 5X_1 - 2X_2$.

[7] (a) Find $E[Y_1]$, $E[Y_2]$, $Var(Y_1)$, $Var(Y_2)$ and $Cov(Y_1, Y_2)$. Are Y_1 and Y_2 independent?

[3] (b) If X_1 and X_2 have bivariate normal distribution, state the joint distribution of Y_1 and Y_2 (including any parameters).

[5] 8. (a) Let $X \sim Geometric\left(\frac{1}{2}\right)$. Use Markov's inequality to get an upper bound for $P(X \geq N)$ where N is a positive integer. Also find the exact probability.

[3] (b) Let $X \sim Binomial(n, p)$. Use Chebyshev's inequality to get an upper bound for $P\left(\left|\frac{X}{n} - p\right| \geq \frac{p}{10}\right)$.

9. Consider the random sum $Y = \sum_{j=1}^{N} X_j$, where $X_1, X_2, \ldots$ is a sequence of i.i.d. random variables each with the same distribution as X and N is a non-negative integer-valued random variable which is independent of the $X's$.

[5] (a) Derive $E[Y]$ and $Var(Y)$ in terms of the means and variances of X and N. You may assume the conditional expectation results that $E[Y] = E[E[Y|N]]$ and $Var(Y) = E[Var(Y|N)] + Var(E[Y|N])$.

[5] (b) The number of customers N going to a store in a day has $E[N] = 500$ and $Var(N) = 25$. The amount of money X spent by a customer in a day has $E[X] = \mu$ and $Var(X) = \sigma^2$. This is independent of other customers and of N. Let Y be the total receipts for the store in the day. Find $E[Y]$ and $Var(Y)$.

10. Suppose that random variables X and Y are jointly continuous with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-(x+y)} & \text{if } 0 < x < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

[5] (a) Find the joint probability density function for $U = Y - X$ and $V = X$.

[7] (b) Show that U and V are independent and find their marginal density functions. State the distributions, means and variances for U and V.

[3] (c) Hence or otherwise find $Cov(X, Y)$.

[5] 11. (a) Z has double exponential distribution with probability density function $f_Z(z) = \frac{\theta}{2} e^{-\theta|z|}$ for $-\infty < z < \infty$. Show that the moment generating function $M_Z(t) = \left(1 - \frac{t^2}{\theta^2}\right)^{-1}$ for $|t| < \theta$.

[10] (b) Let X and Y be independent exponential random variables with patrameter θ, so that $M_X(t) = M_Y(t) = \left(1 - \frac{t}{\theta}\right)^{-1}$.

i. Find the moment generating function of $U = X + Y$ and hence state the distribution of U.
ii. Find the moment generating function of $V = X - Y$ and hence state the distribution of V.
iii. By using the joint moment generating function or otherwise, show that U and V are not independent.

End of examination