MTH5118 Probability II Test Solutions 2008

1. X and Y are independent random variables with $X \sim Bernoulli\left(\frac{1}{3}\right)$ and $Y \sim Binomial\left(2, \frac{1}{2}\right)$.

   (a) (6 points) Write down the probability generating functions $G_X(t)$ and $G_Y(t)$.

   $G_X(t) = \left(\frac{1}{3}t + \frac{2}{3}\right)$ and $G_Y(t) = \left(\frac{1}{2}t + \frac{1}{2}\right)^2$

   (b) (4 points) Obtain the probability generating function of $Z = X + Y$.

   $G_Z(t) = G_X(t)G_Y(t) = \left(\frac{1}{3}t + \frac{2}{3}\right)\left(\frac{1}{2}t + \frac{1}{2}\right)^2$

   (c) (10 points) Obtain the probability mass function for Z.

   $G_Z(t) = \frac{1}{12}(t+2)(t^2+2t+1) = \frac{1}{12}(t^3+4t^2+5t+2)$. So picking off coefficients of powers of t gives:

   $P(Z=0) = \frac{1}{6}$, $P(Z=1) = \frac{5}{12}$, $P(Z=2) = \frac{1}{3}$, $P(Z=3) = \frac{1}{12}$ (and $P(Z=z) = 0$ for all other values of z).

2. Let X be a random variable with probability generating function

$$G_X(t) = te^{2(t-1)}$$

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   (a) (12 points) Differentiate the p.g.f. to obtain $E[X]$ and $Var(X)$.

   $G_X^{'}(t) = e^{2(t-1)} + 2te^{2(t-1)} = (1+2t)e^{2(t-1)}$

   $G_X^{''}(t) = 2(1+2t)e^{2(t-1)} + 2e^{2(t-1)} = 4(1+t)e^{2(t-1)}$

   Therefore $E[X] = G_X^{'}(1) = 3$ and $E[X(X-1)] = G_X^{''}(1) = 8$ and so $Var(X) = E[X(X-1)] + E[X] - (E[X])^2 = 8+3-9 = 2$.

   (b) (6 points) Find $P(X=0)$, $P(X=1)$ and $P(X=2)$.

   $P(X=x) = \frac{G_X^{(x)}(0)}{x!}$ and so $P(X=0) = G_X(0) = 0$, $P(X=1) = G_X^{'}(0) = e^{-2}$ and $P(X=2) = \frac{G_X^{''}(0)}{2} = 2e^{-2}$

3. (12 points) Gonzo has £20 to gamble at a casino on a roulette wheel which has one zero. At each game he bets £5 on 'red'. The probability of winning is $\frac{18}{37}$ for each game. If he wins a game he gets his £5 stake back plus an additional £5. If he loses he receives nothing (his stake is lost). He stops playing when he either goes broke (has £0) or reaches £50. Calculate the probability that he goes broke.

   1 unit is £5, so that you want $L_4(0,10)$. Since $p \neq q$, $L_k(M,N) = \frac{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^k}{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^M}$.

   Therefore the probability that Gonzo goes broke is

$$L_4(0,10) = \frac{\left(\frac{19}{18}\right)^{10} - \left(\frac{19}{18}\right)^4}{\left(\frac{19}{18}\right)^{10} - 1}$$

4. (12 points) State the theorem of total probability for expectations. John plays a series of independent games. At each game he pays £1 and tosses three fair coins. If k heads are obtained then he receives £k. The game stops when he gets no heads. Let X be the amount of his gain by the end of the series. Find $E[X]$.

   If $B_1, B_2, B_3, \ldots$ is a partition of the sample space S and X is a random variable defined on S then $E[X] = \sum_j E[X|B_j]P(B_j)$

   Let B_1, B_2, B_3, B_4 be the events that at the first game he has 1, 2, 3 and 0 heads respectively. This is a partition.

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\begin{aligned}
E[X] &= \sum_{j=1}^{4} E[X|B_j]P(B_j) \\
&= (-1+1+E[X])\frac{3}{8} + (-1+2+E[X])\frac{3}{8} + (-1+3+E[X])\frac{1}{8} + (-1)\frac{1}{8} \\
&= \frac{1}{2} + \frac{7}{8}E[X]
\end{aligned}
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   Therefore $\frac{1}{8}E[X] = \frac{1}{2}$ and so $E[X] = 4$.

5.

   (a) (12 points) The number of spam messages, Y, arriving at a spam filter in a day has mean μ. Each message has probability p of being detected by the filter, independently of all other messages. Let X be the number of spam messages detected by the filter. State the conditional distribution of $X|Y = y$ and $E[X|Y = y]$. Hence find $E[X]$.

   $X|Y = y \sim Binomial(y, p)$. So $E[X|Y = y] = yp$. Therefore $E[X] = E[E[X|Y]] = E[Yp] = pE[Y] = p\mu$.

   (b) (12 points) A branching process begins with a single ancestor forming generation zero. In each generation the number of offspring X of an individual has p.m.f. $P(X = 0) = \frac{1}{4}$, $P(X = 1) = \frac{1}{4}$ and $P(X = 2) = \frac{1}{2}$. Find the probability θ that the population will eventually die out.

   θ is the smallest positive root of $G_X(t) = t$. Here $G_X(t) = \frac{1}{4} + \frac{1}{4}t + \frac{1}{2}t^2$. So we solve $2t^2 - 3t + 1 = 0$. This is simple to solve as we know that $t = 1$ is a root. We have $(t-1)(2t-1) = 0$ giving solutions $t = 1$ and $t = \frac{1}{2}$ and so $\theta = \frac{1}{2}$.

   State the probability of eventual extinction if initially there are k ancestors forming generation zero.

   $\theta^k = \left(\frac{1}{2}\right)^k$

6.

   (a) (8 points) A random variable X has p.d.f. $f_X(x) = \theta e^{-\theta x}$ for $x > 0$. The p.d.f. is zero elsewhere. State the range of t for which the moment generating function, $M_X(t)$, exists. Show that $M_X(t) = \left(1 - \frac{t}{\theta}\right)^{-1}$ for this range of values of t.

$M_X(t) = \int_0^\infty e^{tx}\theta e^{-\theta x}dx = \int_0^\infty \theta e^{-(\theta-t)x}dx$ which will only be finite if $t < \theta$. For this range integrating we obtain

$$M_X(t) = \left[-\frac{\theta}{\theta-t}e^{-(\theta-t)x}\right]_{x=0}^{x=\infty} = \frac{\theta}{\theta-t} = \left(1-\frac{t}{\theta}\right)^{-1}$$

(b) (6 points) Use the moment generating function to find $E[X]$.

Either: Use the result that $E[X] = M_X'(0)$. We have $M_X'(t) = \frac{1}{\theta}\left(1-\frac{t}{\theta}\right)^{-2}$ so that $E[X] = M_X'(0) = \frac{1}{\theta}$.

Or: Use the result that $\frac{E[X^r]}{r!}$ is the coefficient or t^r in the power series expansion of $M_X(t)$. Here $M_X(t) = 1 + \frac{t}{\theta} + \frac{t^2}{\theta^2} + ...$ so that $E[X] = \frac{1}{\theta}$