Probability II. Solutions to Problem Sheet 9.

1. $M_X(s) = M_{X,Y}(s,0) = \frac{1}{(1-s)(1-s)} = (1-s)^{-2}$. This is the m.g.f. of $Gamma(1,2)$. Hence, by the uniqueness result for m.g.f.'s, $X \sim Gamma(1,2)$. Therefore $E[X] = 2$ and $Var(X) = 2$.

$M_Y(t) = M_{X,Y}(0,t) = \frac{1}{(1-t)} = (1-t)^{-1}$. This is the m.g.f. for $Exp(1)$. Hence, by the uniqueness result for m.g.f.'s, $Y \sim Exp(1)$. Therefore $E[Y] = 1$ and $Var(Y) = 1$.

$$\frac{\partial^2 M_{X,Y}(s,t)}{\partial s \partial t} = \frac{\partial}{\partial s}\left((1-s)^{-1}(1-s-t)^{-2}\right) = (1-s)^{-2}(1-s-t)^{-2} + 2(1-s)^{-1}(1-s-t)^{-3}$$

If we take $s = t = 0$ in this second derivative we get $E[XY] = 1 + 2 = 3$. Hence $Cov(X,Y) = E[XY] - E[X]E[Y] = 3 - (2)(1) = 1$. Hence $\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{1}{\sqrt{2}}$.

If $U = X - Y$ and $V = Y$, then

$$\begin{aligned} M_{U,V}(s,t) &= E[e^{s(X-Y)+tY}] = E[e^{sX+(t-s)Y}] = M_{X,Y}(s, t-s) \\ &= \frac{1}{(1-s)(1-s-(t-s))} = \frac{1}{(1-s)}\frac{1}{(1-t)} \end{aligned}$$

Either: Since the joint m.g.f. splits into a function of s times a function of t, U and V are independent. Also $M_U(s) = M_{U,V}(s,0) = (1-s)^{-1}$ and $M_V(t) = M_{U,V}(0,t) = (1-t)^{-1}$. Hence (by the uniqueness of the m.g.f.) $U \sim Exp(1)$ and $V \sim Exp(1)$.

Or: Then $M_U(s) = M_{U,V}(s,0) = (1-s)^{-1}$ and $M_V(t) = M_{U,V}(0,t) = (1-t)^{-1}$. Hence $M_{U,V}(s,t) = M_U(s)M_V(t)$ and (by the uniqueness of the joint m.g.f.) $U \sim Exp(1)$ independent of $V \sim Exp(1)$.

2.(a) $E[U] = \mu + \sigma E[X] = \mu$, $Var(U) = \sigma^2 Var(X) = \sigma^2$, $E[V] = \eta + \tau(\rho E[X] + \sqrt{1-\rho^2}E[Y]) = \eta$, $Var(V) = \tau^2\left(\rho^2 Var(X) + (1-\rho^2)Var(Y)\right) = \tau^2$ and $Cov(U,V) = \sigma\tau\rho Var(X) = \sigma\tau\rho$. Therefore the coefficient of correlation is $\frac{Cov(U,V)}{\sqrt{Var(U)Var(V)}} = \frac{\sigma\tau\rho}{\sqrt{\sigma^2\tau^2}} = \rho$.

(b)

$$\begin{aligned} M_{U,V}(s,t) &= E\left[e^{s(\mu+\sigma X)+t(\eta+\tau(\rho X+\sqrt{(1-\rho^2)}Y))}\right] = e^{\mu s+\eta t}E\left[e^{(s\sigma+\rho\tau t)X}\right]E\left[e^{(\sqrt{(1-\rho^2)}\tau t)Y}\right] \\ &= e^{\mu s+\eta t}M_X(\sigma s+\rho\tau t)M_Y(\sqrt{(1-\rho^2)}\tau t) = e^{\mu s+\eta t}e^{(\sigma s+\rho\tau t)^2/2}e^{(\sqrt{(1-\rho^2)}\tau t)^2/2} \\ &= e^{(\mu s+\eta t)+\frac{1}{2}(\sigma^2 s^2+2\rho\sigma\tau st+\tau^2t^2)} \end{aligned}$$

(i) $M_U(s) = M_{U,V}(s,0) = e^{\mu s+\sigma^2 s^2/2}$. This is the m.g.f. of a normal, hence by uniqueness of the m.g.f., $U \sim N(\mu, \sigma^2)$.

(ii) $M_V(t) = M_{U,V}(0,t) = e^{\eta t+\tau^2 t^2/2}$. This is the m.g.f. of a normal, hence by uniqueness of the m.g.f., $V \sim N(\eta, \tau^2)$.

(c) $f_{X,Y}(x,y) = \frac{1}{2\pi} e^{(x^2+y^2)/2}$ for all x, y. Now $U = \mu + \sigma X$ and $V = \eta + \tau(\rho X + \sqrt{(1-\rho^2)}Y)$. The inverses are $X = \left(\frac{U-\mu}{\sigma}\right)$ and $Y = \frac{1}{\sqrt{(1-\rho^2)}}\left(\left(\frac{V-\eta}{\tau}\right) - \rho\left(\frac{U-\mu}{\sigma}\right)\right)$. There are no restrictions on the ranges. Hence for all u, v,

$$\begin{aligned} f_{U,V}(u,v) &= \frac{1}{2\pi} e^{\frac{1}{2}\left[\left(\frac{u-\mu}{\sigma}\right)^2 + \frac{1}{(1-\rho^2)}\left(\left(\frac{v-\eta}{\tau}\right)-\rho\left(\frac{u-\mu}{\sigma}\right)\right)^2\right]} \times \left\| \begin{matrix} \frac{1}{\sigma} & 0 \\ \frac{-\rho}{\sigma\sqrt{1-\rho^2}} & \frac{1}{\tau\sqrt{1-\rho^2}} \end{matrix} \right\| \\ &= \frac{1}{2\pi\sigma\tau\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[(1-\rho^2)\left(\frac{u-\mu}{\sigma}\right)^2 + \left(\frac{v-\eta}{\tau}\right)^2 - 2\rho\left(\frac{u-\mu}{\sigma}\right)\left(\frac{v-\eta}{\tau}\right) + \rho^2\left(\frac{u-\mu}{\sigma}\right)^2\right]} \\ &= \frac{1}{2\pi\sigma\tau\sqrt{(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{u-\mu}{\sigma}\right)^2 - 2\rho\left(\frac{u-\mu}{\sigma}\right)\left(\frac{v-\eta}{\tau}\right) + \left(\frac{v-\eta}{\tau}\right)^2\right]} \end{aligned}$$

(d) Setting $\rho = 0$ in the joint p.d.f. for U, V we obtain

$$f_{U,V}(u,v) = \left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2}\right)\left(\frac{1}{\sqrt{2\pi}\tau} e^{-\frac{1}{2}\left(\frac{v-\eta}{\tau}\right)^2}\right)$$

for all u, v. Since the ranges are independent and the joint p.d.f. splits into a function of u times a function of v, U and V are independent. (Note that the p.d.f. for U is given in the first bracket and the p.d.f. for V is given in the second bracket.)

(e) $f_U(u) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2}$ for all u, hence for all u

$$\begin{aligned} f_{V|U}(v|u) &= \frac{f_{U,V}(u,v)}{f_U(u)} = \frac{1}{\sqrt{2\pi}\tau\sqrt{(1-\rho)^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{u-\mu}{\sigma}\right)^2 - 2\rho\left(\frac{u-\mu}{\sigma}\right)\left(\frac{v-\eta}{\tau}\right) + \left(\frac{v-\eta}{\tau}\right)^2 - (1-\rho^2)\left(\frac{u-\mu}{\sigma}\right)^2\right]} \\ &= \frac{1}{\sqrt{2\pi}\tau\sqrt{(1-\rho)^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\rho^2\left(\frac{u-\mu}{\sigma}\right)^2 - 2\rho\left(\frac{u-\mu}{\sigma}\right)\left(\frac{v-\eta}{\tau}\right) + \left(\frac{v-\eta}{\tau}\right)^2\right]} \\ &= \frac{1}{\sqrt{2\pi}\tau\sqrt{(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{v-\eta}{\tau}\right) - \rho\left(\frac{u-\mu}{\sigma}\right)\right]^2} \\ &= \frac{1}{\sqrt{2\pi}\tau\sqrt{(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)\tau^2}\left[v-\eta-\frac{\rho\tau}{\sigma}(u-\mu)\right]^2} \end{aligned}$$

for all v. Therefore $V|U = u$ has normal distribution with mean $\eta + \rho\frac{\tau}{\sigma}(u-\mu)$ and variance $\tau^2(1-\rho^2)$.

Note: You can see why this result holds from the way U and V were constructed. Conditioning on $U = u$ is the same as fixing $X = (u-\mu)/\sigma$. Then for this fixed value u, $V = \eta + \rho\frac{\tau}{\sigma}(u-\mu) + \tau\sqrt{1-\rho^2}Y = a + bY$ for constants $a = \eta + \rho\frac{\tau}{\sigma}(u-\mu)$ and $b = \tau\sqrt{1-\rho^2}$. Since a linear function of a normal r.v. is normal, This just gives $V|U = u$ has normal distribution with mean $a = \eta + \rho\frac{\tau}{\sigma}(u-\mu)$ and variance $b^2 = \tau^2(1-\rho^2)$.

3. You can integrate either over x first then y or over y first then x. I have done the former.

$$\begin{aligned}
f_Z(z) &= \int_0^z \left[\int_0^y 6\theta^3 e^{-\theta(x+y+z)}dx\right] dy = \int_0^z 6\theta^2 e^{-\theta(y+z)}\left(1-e^{-\theta y}\right) dy \\
&= \int_0^z 6\theta^2 e^{-\theta z}\left(e^{-\theta y}-e^{-2\theta y}\right) dy = 6\theta e^{-\theta z}\left(1-e^{-\theta z}\right) - 3\theta e^{-\theta z}\left(1-e^{-2\theta z}\right) \\
&= 3\theta e^{-\theta z}\left(1-e^{-\theta z}\right)^2
\end{aligned}$$

The range is $0 < z < \infty$.

Hence, for $0 < z < \infty$, the conditional joint p.d.f. for $X, Y|Z = z$ is

$$f_{X,Y|Z}(x,y|z) = \frac{6\theta^3 e^{-\theta(x+y+z)}}{3\theta e^{-\theta z}\left(1-e^{-\theta z}\right)^2} = \frac{2\theta^2 e^{-\theta(x+y)}}{\left(1-e^{-\theta z}\right)^2}$$

for $0 < x < y < z$.

4. $f_{X,Y,Z}(x,y,z) = 6xyze^{-(x+y+z)}$ for $0 < x < y < z < \infty$. Let $U = X$, $V = Y - X$ and $W = Z - Y$. The inverses are $X = U$, $Y = U + V$ and $Z = U + V + W$. Then the ranges $0 < x < y < z < \infty$ become $0 < u < u+v < u+v+w < \infty$ i.e. $u > 0$, $v > 0$ and $w > 0$. Also, for this range,

$$f_{U,V,W}(u,v,w) = 6u(u+v)(u+v+w)e^{-(3u+2v+w)} \times \left\| \begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{array} \right\| = 6u(u+v)(u+v+w)e^{-(3u+2v+w)}$$

The ranges are independent but the joint p.d.f. does not split. Therefore U, V and W are not independent.

5. (i) $f_{X,Y|Z}(x,y|z) = \frac{2}{z^2}$ for $0 < x < y < z$ and $Z \sim Gamma(\theta, 3)$. So the range for the joint p.d.f. of X, Y, Z is $0 < x < y < z < \infty$ and

$$f_{X,Y,Z}(x,y,z) = f_{X,Y|Z}(x,y|z) f_Z(z) = \frac{2}{z^2} \frac{\theta^3 z^2 e^{-\theta z}}{2} = \theta^3 e^{-\theta z}$$

(ii) We integrate $f_{X,Y,Z}(x,y,z)$ over z.

$$f_{X,Y}(x,y) = \int_y^\infty \theta^3 e^{-\theta z} dz = \theta^2 e^{-\theta y}$$

for $0 < x < y < \infty$. Then

$$f_X(x) = \int_x^\infty \theta^2 e^{-\theta y} dy = \theta e^{-\theta x}$$

for $0 < x < \infty$. Hence $X \sim Exp(\theta)$.

$$f_Y(y) = \int_0^y \theta^2 e^{-\theta y} dx = \theta^2 y e^{-\theta y}$$

for $0 < y < \infty$. So $Y \sim Gamma(\theta, 2)$.

(iii) We integrate $f_{X,Y,Z}(x,y,z)$ over y

$$f_{X,Z}(x,z) = \int_x^z \theta^3 e^{-\theta z} dy = (z-x)\theta^3 e^{-\theta z}$$

for $0 < x < z < \infty$. Hence for this range

$$f_{Y|X,Z}(y|x,z) = \frac{f_{X,Y,Z}(x,y,z)}{f_{X,Z}(x,z)} = \frac{\theta^3 e^{-\theta z}}{(z-x)\theta^3 e^{-\theta z}} = \frac{1}{z-x}$$

for $x < y < z$. Hence $Y|X = x, Z = z \sim U(x,z)$.