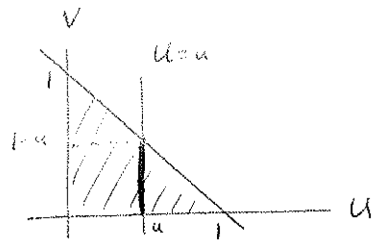
Probability II. Solutions to Problem Sheet 8.

1. $f_{X,Y}(x,y) = 2$ for $0 < x < y < 1$. Let $U = Y - X$ and $V = X$. Inverses are $X = V$ and $Y = U + V$. Then

$$f_{U,V}(u,v) = 2 \times \left\| \begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \right\| = 2$$

The ranges are $0 < v < u + v < 1$, i.e. $v > 0$, $u > 0$ and $u + v < 1$. When $U = u$ then V takes values v where $0 < v < 1 - u$. Note that $0 < u < 1$. Therefore $f_U(u) = \int_0^{1-u} 2dv = 2(1-u)$ for $0 < u < 1$.



2. $f_{X,Y}(x,y) = \frac{\theta^\alpha x^{\alpha-1} e^{-\theta x}}{\Gamma(\alpha)} \frac{\theta^\beta y^{\beta-1} e^{-\theta y}}{\Gamma(\beta)}$ for $x > 0$ and $y > 0$. Let $U = \frac{X}{Y}$ and $V = X + Y$. Inverses are found by substituting $X = YU$ so that $Y = \frac{V}{(1+U)}$ and hence $X = \frac{UV}{(1+U)} \equiv V\left(1 - \frac{1}{(1+U)}\right)$. The latter form is more convenient for differentiation. Then

$$\begin{aligned} f_{U,V}(u,v) &= \frac{\theta^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{uv}{(1+u)}\right)^{\alpha-1} \left(\frac{v}{(1+u)}\right)^{\beta-1} e^{-\theta v} \times \left\| \begin{array}{cc} \frac{v}{(1+u)^2} & \frac{u}{(1+u)} \\ -\frac{v}{(1+u)^2} & \frac{1}{(1+u)} \end{array} \right\| \\ &= \frac{\theta^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} v^{\alpha+\beta-1} e^{-\theta v} \end{aligned}$$

where the ranges are $\frac{uv}{1+u} > 0$ and $\frac{v}{1+u} > 0$, so that $u > 0$ and hence $v > 0$.

Therefore the ranges are independent and the joint p.d.f. splits into a function of u only times a function of v only. Hence U and V are independent and (noticing that $V \sim Gamma(\theta, \alpha + \beta)$),

$$f_V(v) = \frac{\theta^{\alpha+\beta} v^{\alpha+\beta-1} e^{-\theta v}}{\Gamma(\alpha+\beta)}$$

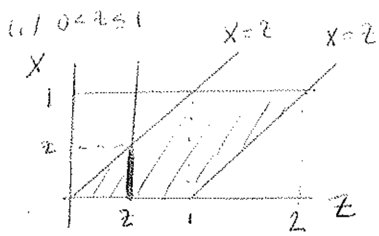
for $v > 0$ and hence, for $u > 0$,

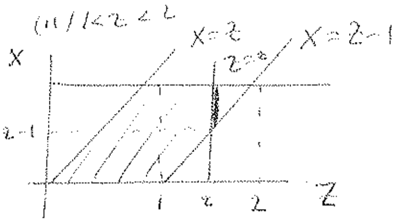
$$f_U(u) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}$$

3. $f_{X,Y}(x,y) = 1$ for $0 < x < 1$ and $0 < y < 1$. Let $Z = X + Y$ and $U = X$. Then $X = U$ and $Y = Z - U$. Hence

$$f_{U,Z}(u,z) = 1 \times \left\| \begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array} \right\| = 1$$

The ranges are $0 < u < 1$ and $0 < z - u < 1$. Since $U \equiv X$ we can write this as $f_{X,Z}(x,z) = 1$ for $0 < x < 1$ and $z - 1 < x < z$.





Either: Consider different ranges of Z, (i) $0 < z \leq 1$; (ii) $1 < z < 2$.

If $0 < z \leq 1$ then $0 < x < z$ and hence $f_Z(z) = \int_0^z dx = z$ and so $f_{X|Z}(x|z) = \frac{1}{z}$ for $0 < x < z$ so that $X|Z = z \sim U(0,z)$ for $0 < z < 1$.

If $1 < z < 2$ then $z - 1 < x < 1$ and hence $f_Z(z) = \int_{z-1}^1 dx = 2 - z$ and so $f_{X|Z}(x|z) = \frac{1}{2-z}$ for $z - 1 < x < 1$ so that $X|Z = z \sim U(z-1, 1)$ for $1 < z < 2$.

Or: Write the ranges as $\max(0, z-1) < x < \min(1, z)$ and $0 < z < 2$.

Then $f_Z(z) = \min(1,z) - \max(0, z-1)$ for $0 < z < 2$ which can be written as $f_Z(z) = z$ for $0 < z \leq 1$ and $f_Z(z) = 2 - z$ for $1 < z < 2$.

Hence $f_{X|Z}(x|z) = \frac{1}{\min(1,z) - \max(0,z-1)}$ for $\max(0, z-1) < x < \min(1,z)$ so that $X|Z = z \sim U(\max(0, z-1), \min(1,z))$. This may also be written as: when $0 < z \leq 1$ then $f_{X|Z}(x|z) = \frac{1}{z}$ for $0 < x < z$ (so $X|Z = z \sim U(0,z)$) and when $1 < z < 2$ then $f_{X|Z}(x|z) = \frac{1}{2-z}$ for $z - 1 < x < 1$ (so $X|Z = z \sim U(z-1, 1)$).

4. (a) $f_U(u) = \theta e^{-\theta(u-\alpha)}$ for $\alpha < u < \infty$. Hence $V = U - \alpha$ has inverse $U = V + \alpha$ and $f_V(v) = \theta e^{-\theta v} \times |1| = \theta e^{-\theta v}$ for $v > 0$ i.e. $V \sim Exp(\theta)$. Therefore $E[U] = E[V + \alpha] = E[V] + \alpha = \frac{1}{\theta} + \alpha$ and $Var(U) = Var(V + \alpha) = Var(V) = \frac{1}{\theta^2}$. (Only statement of result needed)

(b) $f_{X,Y}(x,y) = 2\theta^2 e^{-\theta(x+y)}$ for $0 < x < y < \infty$. Hence

$$f_X(x) = \int_x^\infty 2\theta^2 e^{-\theta(x+y)} dy = \left[-2\theta e^{-\theta(x+y)}\right]_{y=x}^{y=\infty} = 2\theta e^{-2\theta x}$$

for $0 < x < \infty$. Therefore $X \sim Exp(2\theta)$ and so $E[X] = \frac{1}{2\theta}$ and $Var(X) = \frac{1}{4\theta^2}$.

Then for $x > 0$, $f_{Y|X}(y|x) = \frac{2\theta^2 e^{-\theta(x+y)}}{2\theta e^{-2\theta x}} = \theta e^{-\theta(y-x)}$ for $x < y < \infty$.

This is the same form of p.d.f. as in part (a) with $\alpha = x$. Hence $E[Y|X] = \frac{1}{\theta} + X$ and $Var(Y|X) = \frac{1}{\theta^2}$. Therefore

$$E[Y] = E[E[Y|X]] = E\left[\frac{1}{\theta} + X\right] = \frac{1}{\theta} + \frac{1}{2\theta} = \frac{3}{2\theta}$$

$$\begin{aligned} Var(Y) &= E[Var(Y|X)] + Var(E[Y|X]) = E\left[\frac{1}{\theta^2}\right] + Var\left(\frac{1}{\theta} + X\right) \\ &= \frac{1}{\theta^2} + Var(X) = \frac{1}{\theta^2} + \frac{1}{4\theta^2} = \frac{5}{4\theta^2} \end{aligned}$$

$$\begin{aligned} E[XY] &= E[XE[Y|X]] = E\left[X\left(\frac{1}{\theta} + X\right)\right] = \frac{1}{\theta}E[X] + (Var(X) + (E[X])^2) \\ &= \frac{1}{2\theta^2} + \frac{1}{4\theta^2} + \left(\frac{1}{2\theta}\right)^2 = \frac{1}{\theta^2} \end{aligned}$$

Hence $Cov(X,Y) = \frac{1}{\theta^2} - \left(\frac{1}{2\theta}\right)\left(\frac{3}{2\theta}\right) = \frac{1}{4\theta^2}$. Therefore

$$\rho(X,Y) = \frac{\frac{1}{4\theta^2}}{\sqrt{\frac{1}{4\theta^2} \times \frac{5}{4\theta^2}}} = \frac{1}{\sqrt{5}}$$