Probability II. Solution 7.

1.(a) The ranges are independent, but the joint pdf cannot be split into $g(x)h(y)$ for any functions g and h. Therefore X and Y are not independent.

(b) The ranges are independent and $f_{X,Y}(x,y) = C(1+x)(1+y) = g(x)h(y)$, where $g(x) = C(1+x)$ for $0 < x < 1$ and $g(x) = 0$ elsewhere and $h(y) = (1+y)$ for $0 < y < 1$ and $h(y) = 0$ elsewhere. Hence X and Y are independent.

Also $f_X(x) = KC(1+x)$ for $0 < x < 1$ and $f_Y(y) = \frac{1}{K}(1+y)$. Hence, using the result that the p.d.f. must integrate to one, we obtain $1/K = 2/3$ and $CK = 2/3$ so that $K = 3/2$ and $C = 4/9$ and $f_X(x) = \frac{2}{3}(1+x)$ for $0 < x < 1$ and $f_Y(y) = \frac{2}{3}(1+y)$ for $0 < y < 1$.

(c) The ranges for which $f_{X,Y}(x,y) > 0$ are dependent. Hence X and Y are not independent.

(d) The ranges are independent and $f_{X,Y}(x,y) = g(x)h(y)$ where $g(x) = xe^{-x}$ for $0 < x < \infty$ and $g(x) = 0$ elsewhere and $h(y) = \frac{C}{y^5}$ for $1 < y < \infty$ and $h(y) = 0$ elsewhere. Hence X and Y are independent.

Also $f_X(x) = Kxe^{-x}$ for $0 < x < \infty$ and $f_Y(y) = \frac{C}{Ky^5}$ for $1 < y < \infty$. Then $X \sim Gamma(1,2)$ so that $K = 1$. Also $1 = \int_1^\infty \frac{C}{y^5}dy = \left[\frac{-C}{4y^4}\right]_{y=1}^{y=\infty} = \frac{C}{4}$. Hence $C = 4$. Therefore $f_X(x) = xe^{-x}$ for $0 < x < \infty$ and $f_Y(y) = \frac{4}{y^5}$ for $1 < y < \infty$.

2. Note that a chi-squared with parameter n is just a $Gamma(1/2, n/2)$. So $M_X(t) = (1-2t)^{-n/2}$ and $M_Y(t) = (1-2t)^{-m/2}$. Hence $M_Z(t) = M_X(t)M_Y(t) = (1-2t)^{-n/2}(1-2t)^{-m/2} = (1-2t)^{-(n+m)/2}$. Therefore $Z \sim \chi^2_{n+m}$. You could also state this result as $Z \sim Gamma(1/2, (n+m)/2)$.

3. $M_X(t) = \left(1 - \frac{t}{10}\right)^{-1}$ and $M_Y(t) = \left(1 - \frac{t}{10}\right)^{-1}$.

(a) $M_U(t) = M_X(t)M_Y(t) = \left(1 - \frac{t}{10}\right)^{-2}$. Hence $U \sim Gamma(10,2)$.

(b)

$$\begin{aligned} M_V(t) &= E\left[e^{t(X-Y)}\right] = E[e^{tX}]E[e^{-tY}] = M_X(t)M_Y(-t) \\ &= \left(1 - \frac{t}{10}\right)^{-1}\left(1 + \frac{t}{10}\right)^{-1} = \left(1 - \frac{t^2}{100}\right)^{-1} \end{aligned}$$

The double exponential distribution with parameter θ had m.g.f. $\left(1 - \frac{t^2}{\theta^2}\right)^{-1}$. Hence V has double exponential distribution with $\theta = 10$. Therefore the p.d.f. is $f_V(v) = 5e^{-10|v|}$ for $-\infty < v < \infty$.

4. $E[U] = a(E[X]+E[Y]) = a(0+0) = 0$. $Var(U) = a^2(Var(X)+Var(Y)) = a^2(1+1) = 2a^2$. $E[V] = b(E[X] + cE[Y]) = b(0 + c0) = 0$. $Var(V) = b^2(Var(X) + c^2Var(Y)) = b^2(1 + c^2)$. $Cov(U, V) = abVar(X) + abcVar(Y) = ab(1 + c)$.

Hence we need $c = -1$ and $a = b = \frac{1}{\sqrt{2}}$.

Note that $M_X(t) = M_Y(t) = e^{t^2/2}$.

$$M_U(t) = E\left[e^{ta(X+Y)}\right] = E\left[e^{atX}e^{atY)}\right] = M_X(at)M_Y(at) = e^{a^2t^2/2}e^{a^2t^2/2} = e^{t^2/2}$$

This is the m.g.f. for an $N(0, 1)$ distribution. So by the uniqueness of the m.g.f. $U \sim N(0, 1)$.

$$M_V(t) = E\left[e^{tb(X-Y)}\right] = E\left[e^{btX}e^{-btY)}\right] = M_X(bt)M_Y(-bt) = e^{b^2t^2/2}e^{b^2t^2/2} = e^{t^2/2}$$

This is the m.g.f. for an $N(0, 1)$ distribution. So by the uniqueness of the m.g.f. $V \sim N(0, 1)$.