Probability II. Solutions to Problem Sheet 6.

1. (a) $X \sim Exp(\theta)$, hence $f_X(x) = \theta e^{-\theta x}$ for $x > 0$.

Now $Y = 1 - e^{-\theta X} = g(X)$, so the inverse is $X = -\frac{1}{\theta} ln(1 - Y)$. The range of X for which the p.d.f. is positive is $0 < x < \infty$. The corresponding range for Y is just $0 < y < 1$. Hence for $0 < y < 1$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right| = \theta(1-y) \times \left| \frac{1}{\theta(1-y)} \right| = 1$$

The p.d.f. for Y is zero elsewhere. Hence $Y \sim U(0, 1)$.

(b) $X \sim N(0, 1)$. $Y = |X|$, so that Y takes values on $[0, \infty)$. Therefore $F_Y(y) = 0$ for $y \leq 0$.

For $y > 0$, the event $Y \leq y$ is just the event $|X| \leq y$, i.e. $-y \leq X \leq y$.

Hence, for $y > 0$, $F_Y(y) = P(-y < X < y) = F_X(y) - F_X(-y)$.

The p.d.f. will be zero for $y < 0$. For $y > 0$ we can differentiate the c.d.f. above to obtain

$$f_Y(y) = f_X(y) - (-f_X(-y)) = f_X(y) + f_X(-y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}} + \frac{e^{-y^2/2}}{\sqrt{2\pi}} = \left(\sqrt{\frac{2}{\pi}} \right) e^{-y^2/2}$$

2 Since the area of the sample space S is $\frac{1}{2}$ we have $C \times \frac{1}{2} = 1$ and hence $C = 2$.

(i) For $0 < x < 1$, the p.d.f. $f_X(x)$ is just C times the length of the part of the line $X = x$ which lies within the sample space S so that $f_X(x) = 2(1 - x)$ for $0 < x < 1$ and $f_X(x) = 0$ elsewhere.

(ii) For $0 < y < 1$, the p.d.f. $f_Y(y)$ is just C times the length of the part of the line $Y = y$ which lies within the sample space S so that $f_Y(y) = 2(1 - y)$ for $0 < y < 1$ and $f_Y(y) = 0$ elsewhere.

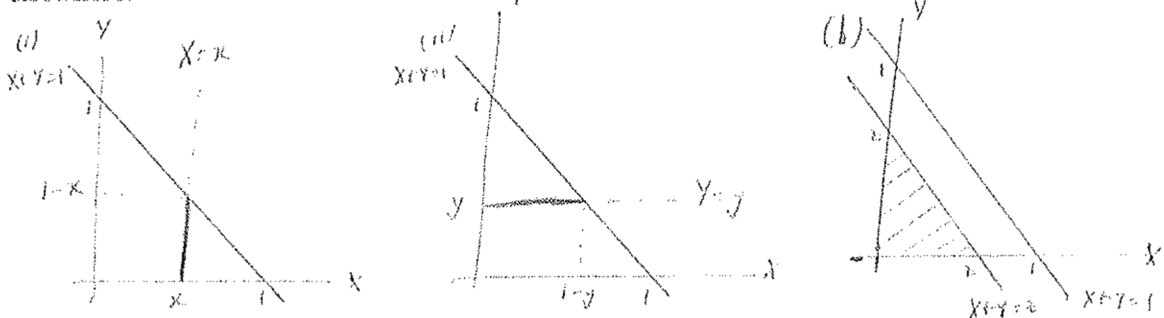
(a) $P(Z \leq z) = P(X + Y \leq z) = 0$ if $z \leq 0$.

(b) If $0 < z < 1$ then $P(Z \leq z) = P(X + Y \leq z)$ where the event $X + Y \leq z$ is just the triangle between the lines $X = 0$, $Y = 0$ and $X + Y = z$. Since the joint p.d.f. is constant

$P(X+Y \le z)$ is just C times the area of this triangle so that $P(Z \le z) = P(X+Y \le z) = 2 \times \frac{z^2}{2} = z^2$.

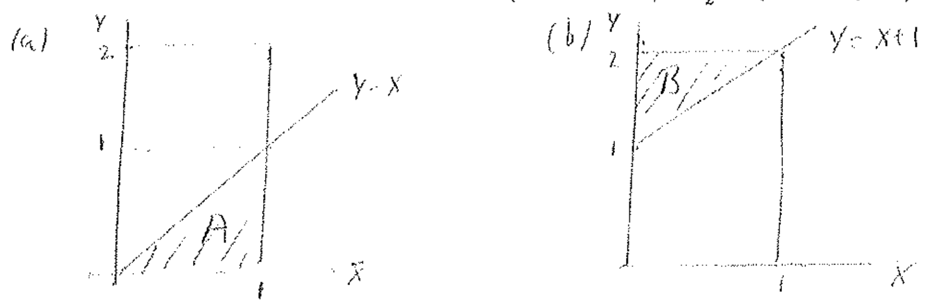
(c) If $z \ge 1$ then $P(Z \le z) = 1$.

Therefore, differentiating the c.d.f. $F_Z(z)$ gives $f_Z(z) = 2z$ for $0 < z < 1$ and $f_Z(z) = 0$ elsewhere.



3. (a) The event that the woman misses the bus is just the event $X > Y$. This is the set A in the diagram which is the interior of the triangle bounded by the X-axis, the lines $X = 1$ and $Y = X$. As the joint p.d.f. is constant $P(X > Y) = \frac{1}{2} \times (Area\ of\ A) = \frac{1}{4}$.

(b) The probability that the woman catches the bus but has to wait at least 1 hour for it to arrive is just the event $Y > X + 1$, which is the set B bounded by the Y-axis and the lines $Y = 2$ and $Y = X + 1$. Hence $P(Y > X+1) = \frac{1}{2} \times (Area\ of\ B) = \frac{1}{4}$.



4. Since the joint p.d.f. integrates to 1,

$$1 = \int_0^2 \left[\int_0^1 Cx(1+y)dx\right] dy = \int_0^2 \left[C(1+y)\frac{x^2}{2}\right]_{x=0}^{x=1} dy = \int_0^2 \frac{C}{2}(1+y)dy = \left[\frac{C}{2}\frac{(1+y)^2}{2}\right]_{y=0}^{y=2} = 2C$$

Therefore $C = \frac{1}{2}$.

$$f_X(x) = \int_0^2 \frac{x}{2}(1+y)dy = \left[\frac{x}{2}\frac{(1+y)^2}{2}\right]_{y=0}^{y=2} = 2x$$

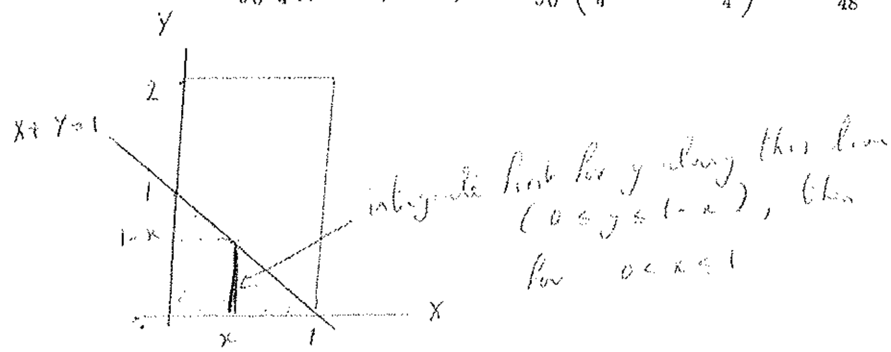
for $0 < x < 1$.

$$f_Y(y) = \int_0^1 \frac{(1+y)}{2} x dx = \left[\frac{(1+y)}{2}\frac{x^2}{2}\right]_{x=0}^{x=1} = \frac{1}{4}(1+y)$$

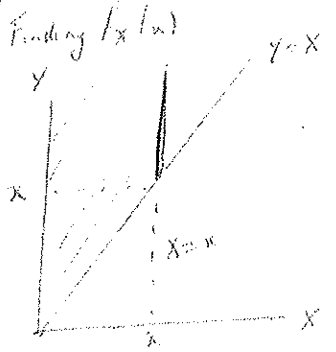
for $0 < y < 2$.

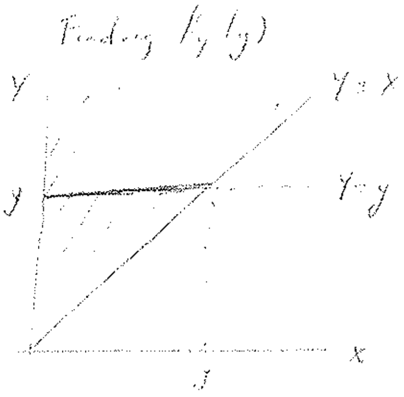
$P(X + Y \leq 1)$ is the probability (X, Y) lies in the area bounded by the axes and the line $X + Y = 1$ and can be obtained by integrating first over y for x fixed and then over x.

$$\begin{aligned} P(X+Y \leq 1) &= \int_0^1 \left[\int_0^{(1-x)} \tfrac{1}{2}x(1+y)dy\right] dx = \int_0^1 \left[\tfrac{x}{2}\tfrac{(1+y)^2}{2}\right]_{y=0}^{y=1-x} dx \\ &= \int_0^1 \tfrac{x}{4}((2-x)^2 - 1)dx = \int_0^1 \left(\tfrac{3x}{4} - x^2 + \tfrac{x^3}{4}\right) dx = \tfrac{5}{48} \end{aligned}$$



5. (a)





$$1 = \int_0^\infty \left[\int_x^\infty Ce^{-y}dy\right] dx = \int_0^\infty \left[-Ce^{-y}\right]_{y=x}^{y=\infty} dx = \int_0^\infty Ce^{-x}dx = C$$

Therefore $C = 1$.

$$f_X(x) = \int_x^{\infty} e^{-y} dy = e^{-x}$$

for $0 < x < \infty$.

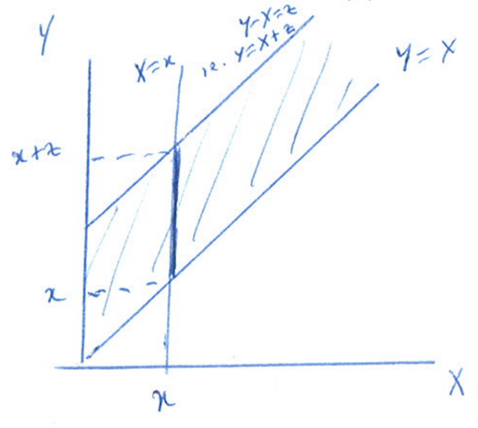
$$f_Y(y) = \int_0^{y} e^{-y} dx = \left[x e^{-y} \right]_{x=0}^{x=y} = y e^{-y}$$

for $0 < y < \infty$.

(b) For $z > 0$, The event $Y - X \leq z$ is the region bounded by the Y-axis and the lines $Y = X$ and $Y = X + z$. Hence

$$\begin{aligned} P(Z \leq z) &= P(Y - X \leq z) \\ &= \int_0^{\infty} \left[\int_x^{x+z} e^{-y} dy \right] dx \\ &= \int_0^{\infty} \left[-e^{-y} \right]_{y=x}^{y=x+z} dx \\ &= \int_0^{\infty} \left(e^{-x} - e^{-(x+z)} \right) dx = 1 - e^{-z}. \end{aligned}$$

Since $F_Z(z) = 0$ for $z \leq 0$ and $F_Z(z) = 1 - e^{-z}$ for $z > 0$, differentiating we obtain $f_Z(z) = e^{-z}$ for $0 < z < \infty$ and $f_Z(z) = 0$ elsewhere.



Integrate first for y from x to (x+z), then for x from 0 to infinity