Probability II. Solution to Problem Sheet 3.

1. This is the standard expected duration of the game for the gambler's ruin problem with $p = 1/2$, $k = 100$ and boundaries $M = 0$ and $N = 200$.

$E[X] = E_{100}(0, 200) = (100 - 0)(200 - 100) = 10,000$

To obtain $E[X]$ if he bets £10 we need to change units so that 1 new unit is equal to £10. So we divide k, N and M by 10 to convert to new units. Hence $E[X] = E_{10}(0, 20) = (10 - 0)(20 - 10) = 100$.

2. Let B_1, B_2 and B_3 correspond to the events that he wins, loses or draws the first game. Let X_k be the number of games he plays starting from k units and let $E_k = E[X_k]$.

$$\begin{aligned} E_k &= E[X_k] = E[X_k|B_1]P(B_1) + E[X_k|B_2]P(B_2) + E[X_k|B_3]P(B_3) \\ &= \frac{1}{3}(1 + E_{k+1}) + \frac{1}{3}(1 + E_{k-1}) + \frac{1}{3}(1 + E_k) \end{aligned}$$

Re-arranging gives $E_{k+1} - 2E_k + E_{k-1} = -3$

The associated quadratic is $\theta^2 - 2\theta + 1 = 0$ with roots both $\theta = 1$ so that the solution to the general equation $E_{k+1} - 2E_k + E_{k-1} = 0$ is $E_k = A + Bk$.

As in lectures consider a particular solution to the actual difference equation of the form $E_k = Ck^2$. Then $C(k+1)^2 - 2Ck^2 + C(k-1)^2 = -3$ and so $2C = -3$ and therefore $C = -3/2$.

Hence the general solution to the actual difference equation is $E_k = A + Bk - \frac{3}{2}k^2$. Now $0 = E_0 = A$ and $0 = E_N = A + BN - \frac{3}{2}N^2$, hence $A = 0$ and $B = \frac{3}{2}N$. Therefore

$$E[X_k] = E_k = \frac{3}{2}Nk - \frac{3}{2}k^2 = \frac{3}{2}k(N - k)$$

3. Let B_i be the event he chooses route i initially.

(a) $E[X] = E[X|B_1]P(B_1) + E[X|B_2]P(B_2) + E[X|B_3]P(B_3) = (10 + E[X])\frac{1}{3} + (15 + E[X])\frac{1}{3} + (8)\frac{1}{3}$

So re-arranging gives $E[X] = 33$.

(b) $E[X] = E[X|B_1]P(B_1) + E[X|B_2]P(B_2) + E[X|B_3]P(B_3) = (10 + E[X_1])\frac{1}{3} + (15 + E[X_2])\frac{1}{3} + (8)\frac{1}{3} = 11 + \frac{1}{3}(E[X_1] + E[X_2])$

$E[X_1] = E[X_1|B_2]P(B_2) + E[X_1|B_3]P(B_3) = (15 + E[X_2])\frac{1}{2} + (8)\frac{1}{2} = \frac{23}{2} + \frac{1}{2}E[X_2]$

$E[X_2] = E[X_2|B_1]P(B_1) + E[X_2|B_3]P(B_3) = (10 + E[X_1])\frac{1}{2} + (8)\frac{1}{2} = 9 + \frac{1}{2}E[X_1]$

Either: Hence adding these two equations gives $(E[X_1] + E[X_2]) = \frac{41}{2} + \frac{1}{2}(E[X_1] + E[X_2])$ and therefore $(E[X_1] + E[X_2]) = 41$. Then

$$E[X] = 11 + \frac{1}{3}(41) = 24\frac{2}{3}$$

Or: You could also find $E[X_1]$ and $E[X_2]$ and then find $E[X]$.

$$E[X_1] = \frac{23}{2} + \frac{1}{2}\left(9 + \frac{1}{2}E[X_1]\right)$$

Therefore $\frac{3}{4}E[X_1] = 16$ and so $E[X_1] = \frac{64}{3}$ and hence $E[X_2] = 9 + \left(\frac{1}{2}\right) \times \left(\frac{64}{3}\right) = \frac{59}{3}$. Hence

$$E[X] = 11 + \frac{1}{3}(E[X_1] + E[X_2]) = 11 + \frac{1}{3}\left(\frac{64}{3} + \frac{59}{3}\right) = 11 + \frac{41}{3} = 24\frac{2}{3}$$

4. When the number of children Y is fixed (as y) the number of daughters X just counts the number of successes (the child is a daughter) out of y independent trials with fixed probability $p = \frac{1}{2}$ of success at each trial. So $X|Y = y$ has binomial distribution parameters $n = y$ and $p = \frac{1}{2}$. Therefore $E[X|Y = y] = \frac{1}{2}y$, $Var(X|Y = y) = \frac{1}{4}y$ and $E[t^X|Y = y] = \left(\frac{1}{2} + \frac{1}{2}t\right)^y$.

(a) $E[X] = E[E[X|Y]] = E[\frac{1}{2}Y] = \frac{1}{2}E[Y] = \frac{1}{2} \times 4 = 2$ and

$$Var(X) = E[Var(X|Y)] + Var(E[X|Y]) = E[Y/4] + Var(Y/2) = \frac{1}{4}E[Y] + \frac{1}{4}Var(Y) = \frac{1}{4}\times 4 + \frac{1}{4}\times 1 = \frac{5}{4}$$

(b) $G_Y(t) = (1 - \theta) + \theta t$. Therefore

$$
\begin{aligned}
G_X(t) &= E[t^X] = E[E[t^X|Y]] = E[\left(\frac{1}{2}+\frac{1}{2}t\right)^Y] \\
&= G_Y\left(\frac{1}{2}+\frac{1}{2}t\right) = (1-\theta)+\theta\left(\frac{1}{2}+\frac{1}{2}t\right) \\
&= \left(1-\frac{\theta}{2}\right)+\frac{\theta}{2}t
\end{aligned}
$$

Therefore, by the uniqueness of the p.g.f. $X \sim Bernoulli\left(\frac{\theta}{2}\right)$.

Alternatively you could say that $P(X=0) = \left(1-\frac{\theta}{2}\right)$ and $P(X=1) = \frac{\theta}{2}$