Probability II. Solutions to Problem Sheet 2.

1. Let B_1, B_2 and B_3 correspond to the events that the sum on the first throw of the dice is respectively 12, 7 and 'neither 12 nor 7'. So $P(B_1) = \frac{1}{36}$, $P(B_2) = \frac{6}{36}$ and $P(B_3) = \frac{29}{36}$. Note that if B_1 occurs then E is certain to occur, if B_2 occurs then E cannot occur and if B_3 occurs then we are in the same situation as at the beginning but now starting the independent trials from trial 2 so that $P(E|B_3) = P(E)$.

Using the theorem of total probability gives

$$P(E) = P(E|B_1)P(B_1) + P(E|B_2)P(B_2) + P(E|B_3)P(B_3) = 1 \times \frac{1}{36} + 0 \times \frac{6}{36} + P(E)\frac{29}{36}$$

Solving gives $P(E) = \frac{1}{7}$.

2 There are 37 equally likely outcomes of which 18 correspond to red so that the probability of winning each game is $p = \frac{18}{37}$. Hence $q/p = (19/18)$. The initial gambling pot is $k = 100$, the upper boundary $N = 200$ and the lower boundary $M = 0$. We want $R_k(M, N) = R_{100}(0, 200)$. This is just

$$\frac{\left(\frac{19}{18}\right)^{100} - 1}{\left(\frac{19}{18}\right)^{200} - 1} = \frac{1}{\left(\frac{19}{18}\right)^{100} + 1}$$

This probability is unaltered if Gonzo has £1000 to gamble and bets £10 each time and stops if he either loses all his money or doubles it. You simply change the unit of money from pounds to units of ten pounds.

3. Now $p = \frac{18}{36} = \frac{1}{2}$. Then

$$R_{100}(0, 200) = \frac{100 - 0}{200 - 0} = \frac{1}{2}.$$

4. Let E_k be the event that Jack loses his money starting from k units and let B_1, B_2 and B_3 be the events that he wins, draws or loses the first game. Let $h_k = P(E_k)$.

$$h_k = P(E_k) = P(E_k|B_1)P(B_1) + P(E_k|B_2)P(B_2) + P(E_k|B_3)P(B_3) = h_{k+1}\frac{1}{3} + h_k\frac{1}{3} + h_{k-1}\frac{1}{3}$$

Hence $h_{k+1} - 2h_k + h_{k-1} = 0$ for $k = 1, ..., N-1$. Also $h_0 = 0$ and $h_N = 0$. The associated quadratic for this difference equation is $z^2 - 2z + 1 = 0$ so that the roots are both 1, The solution is therefore $h_k = A + Bk$. Then $1 = h_0 = A$ and $0 = h_N = A + BN$, so that $B = -\frac{1}{N}$. Therefore $h_k = 1 - \frac{k}{N}$.

So the probability he loses all his money is $1 - \frac{k}{N} = \frac{N-k}{N}$.

Note that you could have obtained this result from the ordinary gambler's ruin result by only considering games where he does not draw. The probability p will then be the probability of winning given that the game is not drawn (so is 1/2 here).

5. The first part is just the gambler's ruin problem with boundaries $M = 0$ and $N = 12$, $p = 1/2$ is the probability of a jump forwards. When $k = 1$ we want the probability he reaches 0, i.e. $L_1(0, 12) = \frac{12-1}{12-0} = \frac{11}{12}$. When $k = 11$ we want the probability that he reaches 12, i.e. $R_{11}(0, 12) = \frac{11-0}{12-0} = \frac{11}{12}$.

Now consider the random walk on a circle with the points labelled clockwise 0 to 11 and $p = 1/2$ is the probability of jumping one place clockwise. Let E be the event he returns to his starting point (0) without making a circuit. Let B_1 be the event that he initially jumps clockwise and B_2 be the event that he initially jumps anti-clockwise.

We will have two labels for the point 0, 0 and 12. We use the label 0 when the frog is at position 1 and jumps anti-clockwise to the point and the label 12 when the frog is at position 11 and jumps clockwise to the point.

Hence if the frog jumps clockwise initially (event B_1) he then needs to reach 0 before reaching 12 if he is not to complete a circuit before reaching his starting point again. So $P(E|B_1)$ is equivalent to the probability in the gambler's ruin problem that the gambler start with 1 and reaches 0 before reaching 12. therefore $P(E|B_1) = L_1(0, 12)$, which we found in the first part to be $\frac{11}{12}$.

If the frog jumps anti-clockwise initially (event B_2) he lands on 11 and $P(E|B_2)$ is the probability that he reaches 12 before reaching 0. Hence $P(E|B_2) = R_{11}(0, 12)$, which we found in the first part to be $\frac{11}{12}$.

Since $P(B_1) = P(B_2) = \frac{1}{2}$ we have

$$P(E) = P(B_1)P(E|B_1) + P(B_2)P(E|B_2) = \frac{1}{2}L_1(0, 12) + \frac{1}{2}R_{11}(0, 12) = \frac{1}{2}\frac{11}{12} + \frac{1}{2}\frac{11}{12} = \frac{11}{12}$$

If $p \neq 1/2$ then for the random walk on the integers we have:

When $k = 1$ we want the probability he reaches 0 before 12 which is now $L_1(0,12) = \frac{\left(\frac{q}{p}\right)^{12} - \left(\frac{q}{p}\right)^{1}}{\left(\frac{q}{p}\right)^{12} - 1}$.

When $k = 11$ we want the probability he reaches 12 before 0 which is now $R_{11}(0,12) = \frac{\left(\frac{q}{p}\right)^{11} - 1}{\left(\frac{q}{p}\right)^{12} - 1}$.

$P(B_1) = p$, $P(B_2) = q$ and $P(E|B_1) = L_1(0,12) = \frac{\left(\frac{q}{p}\right)^{12} - \left(\frac{q}{p}\right)^{1}}{\left(\frac{q}{p}\right)^{12} - 1}$ and $P(E|B_2) = R_{11}(0,12) = \frac{\left(\frac{q}{p}\right)^{11} - 1}{\left(\frac{q}{p}\right)^{12} - 1}$. Therefore

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\begin{aligned}
P(E) &= P(B_1)P(E|B_1) + P(B_2)P(E|B_2) = pL_1(0,12) + qR_{11}(0,12) \\
&= p \times \frac{\left(\frac{q}{p}\right)^{12} - \left(\frac{q}{p}\right)^{1}}{\left(\frac{q}{p}\right)^{12} - 1} + q \times \frac{\left(\frac{q}{p}\right)^{11} - 1}{\left(\frac{q}{p}\right)^{12} - 1} \\
&= \frac{2q\left(\left(\frac{q}{p}\right)^{11} - 1\right)}{\left(\frac{q}{p}\right)^{12} - 1}
\end{aligned}
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