Probability II. Solutions to Sheet 10.

1. (a) Markov's inequality for a non-negative r.v. X with mean μ states that for any $h > 0$, $P(X \geq h) \leq \frac{\mu}{h}$. So here we simply take $h = \mu + 2\sigma$ to obtain

$$P(X \geq \mu + 2\sigma) \leq \frac{\mu}{\mu + 2\sigma}$$

So the upper bound for $P(X \geq \mu + 2\sigma)$ is $\frac{\mu}{\mu+2\sigma}$.

(b) If X has mean μ and variance σ^2 then Chebyshev's inequality states that, for any $h > 0$,

$$P(|X - \mu| \geq h) \leq \frac{\sigma^2}{h^2}$$

So we just need to take $h = 2\sigma$. Then Chebyshev's inequality states that

$$P(|X - \mu)| \geq 2\sigma) \leq \frac{\sigma^2}{(2\sigma)^2} = \frac{1}{4}$$

So the upper bound for $P(|X - \mu| \geq 2\sigma)$ is $\frac{1}{4}$.

If $X \sim Exp(\theta)$, then $\mu = \frac{1}{\theta}$ and $\sigma^2 = \frac{1}{\theta^2}$. Then:

(a) Markov's inequality is just $P\left(X \geq \frac{3}{\theta}\right) \leq \frac{1}{3}$. The exact probability is just

$$P\left(X \geq \frac{3}{\theta}\right) = \int_{\frac{3}{\theta}}^{\infty} \theta e^{-\theta x} dx = e^{-3} = 0.04979$$

(b) Chebyshev's inequality is just $P\left(\left|X - \frac{1}{\theta}\right| \geq \frac{2}{\theta}\right) \leq \frac{1}{4}$ The exact probability is just

$$P\left(\left|X - \frac{1}{\theta}\right| \geq \frac{2}{\theta}\right) = P\left(X \geq \frac{3}{\theta}\right) + P\left(X \leq -\frac{1}{\theta}\right) = \int_{\frac{3}{\theta}}^{\infty} \theta e^{-\theta x} dx = e^{-3} = 0.04979$$

2. $E[\overline{X}_n] = p$ and $Var(\overline{X}_n) = \frac{p(1-p)}{n}$.

(a) Applying Chebyshev's inequality to $\overline{X}_n$, and letting $h = 0.1p$ gives

$$P(|\overline{X}_n - p| \geq 0.1p) \leq \frac{p(1-p)/n}{(0.1p)^2} = \frac{100(1-p)}{np}$$

Hence $P(|\overline{X}_n - p| \geq 0.1p) \leq 0.05$ provided $\frac{100(1-p)}{np} \leq 0.05$, i.e. $n \geq \frac{100(1-p)}{0.05p} = \frac{2000(1-p)}{p}$.

(b) The Central Limit Theorem implies that if $Z = \frac{\sqrt{n}(\overline{X}_n - p)}{\sqrt{p(1-p)}}$, then for n large $P(Z \leq z) \simeq \Phi(z)$ where Φ is the c.d.f. for the $N(0,1)$ distribution. Here

$$P(|\overline{X}_n - p| \geq 0.1p) = P\left(|Z| \geq \frac{\sqrt{n}0.1p}{\sqrt{p(1-p)}}\right) \simeq 2\left(1 - \Phi\left(0.1\sqrt{\frac{np}{(1-p)}}\right)\right)$$

Hence we want $\Phi\left(0.1\sqrt{\frac{np}{(1-p)}}\right) \simeq 0.975$, so $0.1\sqrt{\frac{np}{(1-p)}} = 1.96$ and therefore $n \approxeq \frac{(19.6)^2(1-p)}{p} = 384.16\left(\frac{1}{p} - 1\right)$.

3. Let $\mathbf{X}$ have vector of means $\mathbf{m}$ and variance-covariance matrix $\mathbf{V}$ and let $\mathbf{Y} = \mathbf{AX} + \mathbf{b}$. Then $E[\mathbf{Y}] = \mathbf{Am} + \mathbf{b}$ and $Var(\mathbf{Y}) = \mathbf{AVA}^T$. Here

$$\mathbf{m} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} 9 & -2 \\ -2 & 4 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & -7 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Therefore

$$E[\mathbf{Y}] = \mathbf{Am} + \mathbf{b} = \begin{pmatrix} 1 & 1 \\ 2 & -7 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ -11 \end{pmatrix}$$

and the variance-covariance matrix for $\mathbf{Y}$, $Var(\mathbf{Y})$, is

$$\mathbf{AVA}^T = \begin{pmatrix} 1 & 1 \\ 2 & -7 \end{pmatrix}\begin{pmatrix} 9 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -7 \end{pmatrix}\begin{pmatrix} 7 & 32 \\ 2 & -32 \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 0 & 288 \end{pmatrix}$$

Hence $Cov(Y_1, Y_2) = 0$. Although independence implies covariance zero, a covariance of zero does not imply independence. If the covariance had been non-zero you would know that Y_1 and Y_2 could not be independent. Since here $Cov(Y_1, Y_2) = 0$ you can't tell if they are independent; there is insufficient information.

If X_1 and X_2 are bivariate normal, since Y_1 and Y_2 are linear functions of bivariate normals they are also bivariate normal. Since $Cov(Y_1, Y_2) = 0$ this now implies (from results for bivariate normals) that Y_1 and Y_2 are independent normal. Therefore $Y_1 \sim N(10, 9)$ independent of $Y_2 \sim N(-11, 288)$