MAS228 Probability II

Test Solutions 9^{th} November 2006

1. X and Y are independent random variables with $X \sim Bernoulli(1/2)$ and $Y \sim Bernoulli(1/4)$.

   (a) (4 points) Write down the probability generating functions $G_X(t)$ and $G_Y(t)$.
   $G_X(t) = \frac{1}{2} + \frac{1}{2}t$ and $G_Y(t) = \frac{3}{4} + \frac{1}{4}t$

   (b) (4 points) Obtain the probability generating function of $Z = X + Y$.
   $G_Z(t) = G_X(t)G_Y(t) = \left(\frac{1}{2} + \frac{1}{2}t\right)\left(\frac{3}{4} + \frac{1}{4}t\right)$

   (c) (10 points) Obtain the probability mass function for Z.
   Expand $G_Z(t)$ in powers of t. Then $P(Z = z)$ is the coefficient of t^z in the expansion. $G_Z(t) = \frac{3}{8} + \frac{1}{2}t + \frac{1}{8}t^2$. Hence $P(Z = 0) = \frac{3}{8}$, $P(Z = 1) = \frac{1}{2}$, $P(Z = 2) = \frac{1}{8}$ and $P(Z = z) = 0$ for all other values of z.

2. Let X be a random variable with probability generating function

$$G_X(t) = \frac{1}{2}(1+t)e^{(t-1)}$$

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   (a) (10 points) Differentiate the p.g.f. to obtain $E[X]$ and $Var(X)$.

$$G_X'(t) = \frac{1}{2}e^{(t-1)} + \frac{1}{2}(1+t)e^{(t-1)} = \left(1 + \frac{1}{2}t\right)e^{(t-1)}$$

$$G_X''(t) = \frac{1}{2}e^{(t-1)} + \left(1 + \frac{1}{2}t\right)e^{(t-1)} = \left(\frac{3}{2} + \frac{1}{2}t\right)e^{(t-1)}$$

   Therefore $E[X] = G_X'(1) = \frac{3}{2}$ and $E[X(X-1)] = G_X''(1) = 2$. Therefore $Var(X) = E[X(X-1)] + E[X] - (E[X])^2 = 2 + \frac{3}{2} - \frac{9}{4} = \frac{5}{4}$.

   (b) (6 points) Find $P(X = 0)$, $P(X = 1)$ and $P(X = 2)$.
   $P(X = r) = \frac{1}{r!}G_X^{(r)}(0)$. Therefore $P(X = 0) = G_X(0) = \frac{1}{2}e^{-1}$, $P(X = 1) = G_X'(0) = e^{-1}$ and $P(X = 2) = \frac{1}{2}G_X''(0) = \frac{3}{4}e^{-1}$.

3.

Gonzo has £100. He plays roulette and at each game has the same stake. The probability of winning is p (and of losing is $q = 1 - p$) for each game. If he wins he receives double his stake and if he loses he receives nothing (his stake is lost). He stops playing when he either goes broke (has £0) or reaches £1000.

   (a) (10 points) Give the probability that he goes broke (in terms of q/p) if $p \neq \frac{1}{2}$ and his stake at each game is £1.

   Since we want the probability of reaching the lower boundary and a unit stake is used, we need to find $l_k(M,N)$ for the case $p \neq \frac{1}{2}$ when $k = 100$, $N = 1000$ and $M = 0$. There are different ways of writing $l_k(M,N)$, all of which are equivalent.

$$l_k(M,N) = \frac{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^k}{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^M} = \frac{\left(\frac{q}{p}\right)^{N-M} - \left(\frac{q}{p}\right)^{k-M}}{\left(\frac{q}{p}\right)^{N-M} - 1}$$

Hence the probability Gonzo goes broke is just $l_{10}(0,1000) = \frac{\left(\frac{q}{p}\right)^{1000} - \left(\frac{q}{p}\right)^{10}}{\left(\frac{q}{p}\right)^{1000} - 1}$

Note that you can always check your expression for $l_k(M,N)$. If $k = M$ you should get $l_M(M,N) = 1$. If you get zero you have the wrong boundary (i.e. your expression is for $r_k(M,N)$).

(b) (10 points) Give the probability that he goes broke if $p = \frac{1}{2}$ and his stake at each game is £10.

You now need to use $l_k(M,N)$ for the case $p = \frac{1}{2}$. Since the stake is now £10, you need to change the units so that £10 becomes 1 new unit (i.e. $k = 100/10 = 10$ new units, $N = 1000/10 = 100$ new units and $M = 0/10 = 0$ new units).

Now $l_k(M,N) = \frac{N-k}{N-M}$, so $l_{10}(0,100) = \frac{100-10}{100-0} = \frac{9}{10}$.

Again you can check your expression since $l_M(M,N) = 1$.

For the case $p = \frac{1}{2}$, since the same scaling factor is applied to all of k, M and N you get the same result with and without scaling. However if you calculate $l_{100}(0,1000)$ you must also explain that you need to count in new units so that the stake is one new unit, but that the result is invariant to scaling to new units so that it suffices to calculate the unscaled value.

4. (12 points) A mouse is placed at the centre of a maze with 3 paths. Two paths lead to a dead end so that he retraces the path back to the centre. The times taken (to follow the path and return to the centre) are 10 minutes and 12 minutes respectively. The third path leads out of the maze after 15 minutes. Each time the mouse is at the centre of the maze he chooses one of the three paths at random.

Find $E[T]$ where T is the time in minutes until he gets out of the maze.

Let A_i be the event 'mouse chooses path i'. Then you use the law of total probability for expectations, i.e.

$$E[T] = E[T|A_1]P(A_1) + E[T|A_2]P(A_2) + E[T|A_2]P(A_2)$$

Hence we obtain

$$E[T] = (10 + E[T])\frac{1}{3} + (12 + E[T])\frac{1}{3} + (15)\frac{1}{3} = \frac{37}{3} + \frac{2}{3}E[T]$$

Hence $\frac{1}{3}E[T] = \frac{37}{3}$ and so $E[T] = 37$.

5. A population of amoebae begins with a single individual. In each generation an individual dies with probability $1/4$ (i.e. has no offspring) or has two offspring (by splitting in two) with probability $3/4$.

(a) (10 points) Find the expected number of offspring in generation n.

Let Y_n be the number in generation n. Then if X_j is number of offspring of j^{th} individual of population n, $Y_{n+1} = \sum_{j=1}^{Y_n} X_j$. The X_j are i.i.d. random variables. Hence $E[Y_{n+1}] = E[E[Y_{n+1}|Y_n]] = E[Y_n E[X]] = E[X]E[Y_n]$.
Therefore $E[Y_n] = E[X]E[Y_{n-1}] = ... = (E[X])^n E[Y_0] = (E[X])^n$.
Here $P(X=0) = \frac{1}{4}$ and $P(X=2) = \frac{3}{4}$ so that $E[X] = 0P(X=0) + 2P(X=2) = \frac{3}{2}$.
Therefore the expected number of offspring in generation n is $E[Y_n] = \left(\frac{3}{2}\right)^n$.

(b) (10 points) Find the probability that the population will die out eventually.

The probability of eventual extinction is just the smallest positive root of $G_X(t) = t$. Note that $t = 1$ must be a solution to this equation. $G_X(t) = \frac{1}{4} + \frac{3}{4}t^2$. Hence $G_X(t) = t$ gives $1 + 3t^2 = 4t$ i.e. $3t^2 - 4t + 1 = 0$. since $t = 1$ is a solution is is easy to factorise the quadratic giving $(3t-1)(t-1) = 0$. Hence the roots are $t = 1, \frac{1}{3}$. Therefore the probability of eventual extinction is $\frac{1}{3}$.

6. $X \sim Exp(2)$ so that the p.d.f. is $f_X(x) = 2e^{-2x}$ for $x > 0$. The p.d.f. is zero elsewhere.

(a) (7 points) Show that the moment generating function is $M_X(t) = \left(1 - \frac{t}{2}\right)^{-1}$. State the constraint required on t.

$$M_X(t) = E[e^{tx}] = \int_0^\infty e^{tx} 2e^{-2x} dx$$

Note that if $t \geq 2$ the integral will be infinite. If $t < 2$ then

$$M_X(t) = \frac{2}{2-t} \int_0^\infty (2-t)e^{-(2-t)x} dx$$

The integral is just the integral of the p.d.f. for $Exp(2-t)$ so is equal to one. Hence $M_X(t) = \frac{2}{2-t} = \left(1 - \frac{t}{2}\right)^{-1}$ provided $t < 2$.
Note that you can just do the simple integration. If $t \geq 2$ the evaluation at the upper end-point of the range will be infinite.

(b) (7 points) Obtain $E[X^r]$ for all positive integer values of r.

Since $M_X(t) = \sum_{r=0}^\infty \frac{E[X^r]t^r}{r!}$ the easiest thing to do here is expand $M_X(t)$ in a power series. Then $E[X^r]$ is just $r!$ times the coefficient of t^r in the expansion.

$$M_X(t) = \left(1 - \frac{t}{2}\right)^{-1} = \sum_{r=0}^\infty \left(\frac{t}{2}\right)^r = \sum_{r=0}^\infty t^r \left(\frac{1}{2}\right)^r$$

Hence $E[X^r] = \frac{r!}{2^r}$.
Alternatively you can differentiate the m.g.f. r times with respect to t and put $t = 0$. This gives $E[X^r]$.

$M_X'(t) = \frac{1}{2}\left(1 - \frac{t}{2}\right)^{-2}$, $M_X^{(2)}(t) = 2\left(\frac{1}{2}\right)^2\left(1 - \frac{t}{2}\right)^{-3}$, $M_X^{(3)}(t) = 3!\left(\frac{1}{2}\right)^3\left(1 - \frac{t}{2}\right)^{-4}$. Hence $M_X^{(r)}(t) = r!\left(\frac{1}{2}\right)^r\left(1 - \frac{t}{2}\right)^{-(r+1)}$. Therefore $E[X^r] = \frac{r!}{2^r}$.