Queen Mary, University of London

B.Sc. Examination 2005/2006

MAS228 Probability II

Duration: 2 hours

Date and time:

The paper has two sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Electronic calculators may be used. The make and model should be specified on the script. The calculator must not be preprogrammed (other than by the manufacturer) prior to the examination.

Section A: You should attempt all questions. Marks awarded are shown next to the question. This part of the examination carries 60 % of the marks.

1. Let X be a random variable with probability generating function

$$G_X(t) = e^{-2+2t^2}.$$

[4] (a) Using $G_X(t)$, find $P(X = 1)$ and $P(X = 4)$.

[4] (b) Using $G_X(t)$, find $E(X)$ and $\text{Var}(X)$.

2. A population of bacteria begins with a single individual. In each generation, each individual dies with probability $1/2$ or trebles (splits in three) with probability $1/2$.

[4] (a) Find the probability that the population will die out by generation 3.

[3] (b) Find the probability that the population will eventually die out.

3. A fair die is successively rolled. Let X and Y denote, respectively, the number of rolls necessary to obtain a 6 and a 5.

[4] (a) Find the conditional probability density $p_{X|Y}(x|1)$.

[4] (b) Using your answer to part (a), find $E(X|Y = 1)$.

6 4. Every second a robot either moves two metres forward with probability $2/5$ or two metres backwards with probability $3/5$. Four metres in front of the robot's starting point is a door. Two metres behind the robot's starting point is a hole. What is the probability that the robot lands in the hole before it reaches the door?

5. Suppose that the number of customers entering a particular store on a typical day is Poisson(50) distributed and that each customer buys one item with probability $1/4$ and two items with probability $3/4$ independently of each other. Let X be the number of items bought at the store in a particular day.

3 (a) Find $E(X)$.

4 (b) Find $\mathrm{Var}(X)$.

6. Random variables X and Y are jointly continuous with probability density function

$$f_{X,Y}(x,y) = \begin{cases} C(3x^2y+y) & \text{if } 0 < x < 1 \text{ and } 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

3 (a) Find C.

3 (b) Determine whether X and Y are independent.

4 (c) Find $E(X)$, $E(Y)$, $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$ and $\mathrm{Cov}(X,Y)$.

7. Let X be Binomial$(90, 1/3)$ distributed.

3 (a) Use Markov's inequality to get an upper bound for $P(X \geq 50)$.

4 (b) Use Chebyshev's inequality to get an upper bound for $P(X \geq 50)$.

8. Suppose that X_1 and X_2 are two independent $\mathrm{N}(0,1)$ distributed random variables and let $Y_1 = \alpha X_1 + X_2 + 3$, $Y_2 = -4\alpha X_1 + X_2 - 1$ where α is a constant.

4 (a) Find all α for which Y_1 and Y_2 are independent.

3 (b) For all α found in part (a), find $E(Y_1)$, $E(Y_2)$, $\mathrm{Var}(Y_1)$, $\mathrm{Var}(Y_2)$ and $\mathrm{Corr}(Y_1, Y_2)$.

Turn Over ...

Section B: You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best TWO questions answered will be counted. This part of the examination carries 40% of the marks.

1. (a) Suppose that X and Y are random variables and $Y = aX + b$ for constants a and b. State and prove a relationship between the moment generating functions $M_X(t)$ and $M_Y(t)$.

   (b) Let Y be a random variable with probability density function

$$f_Y(y) = \begin{cases} \frac{1}{4}(y-3)e^{(3-y)/2} & \text{if } y > 3 \\ 0 & \text{f } y \leq 3. \end{cases}$$

      i. Find the moment generating function $M_Y(t)$.

      ii. Using your answer to part (b), find $E(Y)$ and $E(Y^2)$.

      iii. Using part (a) and (b), prove that Y has the distribution of the random variable $aX+b$, where a and b are constants and X is $\Gamma(2,1)$ distributed. Name a and b.

2. Suppose that random variables X and Y are jointly continuous with probability density function

$$f_{X,Y}(x,y) = \begin{cases} 2x & \text{if } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Calculate $P(X - Y \leq 1/2)$.

   (b) Calculate $P(XY \leq 1/4)$.

   (c) Calculate $P(X^2 + Y^2 \leq 1)$.

   (d) Calculate $E(|X - Y|)$.

   (e) Calculate $E(|X^2 - Y|)$.

Turn Over ...

3. (a) Suppose that X_i are independent and identically distributed random variables with common probability generating function $G_X(t)$ and that N is a random variable independent of the X_i with probability generating function $G_N(t)$. State and prove a formula giving the probability generating function of the random variable $S_N = \sum_{i=1}^{N} X_i$ in terms of $G_X(t)$ and $G_N(t)$.

   (b) Given a branching process in which the number of descendents of a particular individual is a random variable X with probability generating function $G(t)$, prove that if u_n is the probability that a branching process has 0 individuals by generation n and u_∞ is defined to be $u_\infty = \lim_{n\to\infty} u_n$, that u_∞ satisfies the equation $u_\infty = G(u_\infty)$.

4. Suppose that X_1 and X_2 are independent Exponential(1) distributed random variables.

   (a) Find the joint probability density function $f_{Y_1,Y_2}(y_1,y_2)$ of $Y_1 = X_1 + X_2$ and $Y_2 = X_1/X_2$.

   (b) Determine whether Y_1 and Y_2 are independent.

End of examination.