Queen Mary
University of London

B.Sc. Examination by course units

MAS228 Probability II

Duration: 2 hours

Date and time: 7th May 2008, 2.30-4.30 pm

The paper has two sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators ARE permitted in this examination, but no programming, graph plotting or algebraic facility may be used. The unauthorised use of material stored in a pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.

You should not start reading this paper until instructed to do so by the invigilator.

Section A: You should attempt all questions. Marks awarded are shown next to the questions.

1. Let X be a non-negative integer-valued random variable which has probability generating function $G_X(t) = \frac{1}{2}\left(1 - \frac{t^2}{2}\right)^{-1}$

   4 (a) Find $E(X)$ and $Var(X)$.

   4 (b) Find $P(X=0)$, $P(X=1)$, $P(X=2)$ and $P(X=4)$.

5 2. A gambler has a pot of £100. He plays a series of games. At each game he has probability $\frac{2}{3}$ of winning and probability $\frac{1}{3}$ of losing. He bets £10 at each game which is lost if he loses; if he wins then he gets his stake back plus £10. He continues playing until either his gambling pot reaches £500 or he loses all his money.

   Calculate the probability that his pot reaches £500.

7 3. State the law of total probability for expectations.

   Gonzo plays a series of games. At each game he throws two coins. If no heads show then he stops. Otherwise he continues playing. Let X count the total number of heads obtained in the series of games. Obtain $E[X]$.

4. The number of female offspring X for a female in a certain isolated society has p.m.f. $P(X=0) = \frac{1}{4}$, $P(X=1) = \frac{1}{4}$ and $P(X=2) = \frac{1}{2}$. There is no immigration or emigration so that the society dies out eventually if the female line of descent dies out.

   3 (a) Find the expected number of females in generation n from one female ancestor in generation zero.

   6 (b) Consider the female line of descent of a specific female. Find the probability that her female line of descent dies out eventually. If initially there are k females in the society, find the probability that the population eventually becomes extinct.

5 5. X and Y are independent $N(0,1)$ random variables with moment generating functions $M_X(t) = M_Y(t) = e^{\frac{1}{2}t^2}$. Let $Z = \frac{X+Y}{\sqrt{2}}$. Find $M_Z(t)$ and state the distribution of Z (including any parameters).

6. Random variables X and Y are jointly continuous with probability density function

$$f_{X,Y}(x,y) = \begin{cases} Cxy^2 & \text{if } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

3 (a) Determine if X and Y are independent.

6 (b) Find C and the marginal density functions for X and Y.

7 7. Let X_1 and X_2 be random variables with $E[X_1] = 7$, $E[X_2] = 5$, $Var(X_1) = 4$, $Var(X_2) = 9$ and $Cov(X_1, X_2) = -1$. Define $Y_1 = X_1 - X_2$ and $Y_2 = 3X_1 + X_2$.

Find $E[Y_1]$, $E[Y_2]$, $Var(Y_1)$, $Var(Y_2)$ and $Cov(Y_1, Y_2)$. Are Y_1 and Y_2 independent?

8. Let X be a random variable which takes non-negative values only and let $E[X] = \mu$ and $Var(X) = \sigma^2$.

6 (a) Use Markov's inequality to get an upper bound for $P(X \geq 3\mu)$. Find the exact probability if $X \sim Exp(\theta)$.

4 (b) Use Chebyshev's inequality to get a lower bound for $P(|X - \mu| < 2\sigma)$.

Section B: Each question carries 20 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best TWO questions answered will be counted.

1. Consider a branching process which starts with a single individual in generation zero. Individuals act independently and the number of offspring X for an individual has the same distribution for all individuals in all generations. Let Y_n be the number of individuals in generation n. Use standard conditional expectation results to prove that $G_{Y_{n+1}}(t) = G_{Y_n}(G_X(t))$ for $n = 1, 2, ...$

   Now suppose that $X \sim Bernoulli(p)$, where $0 < p < 1$.

   (a) Prove that $Y_n \sim Bernoulli(p^n)$.

   (b) Find the probability of extinction by generation n, θ_n. Let T be the generation when the branching process first becomes extinct. By writing θ_n as the probability of an event for T show that T has a geometric distribution and state its parameter.

2. (a) If $X \sim Binomial(n, p)$ independent of $Y \sim Binomial(m, p)$ derive the distribution of $Z = X + Y$. Find $P(X = x, Z = z)$ and hence obtain the conditional distribution of $X|Z = z$.

   (b) X and Y have trinomial distribution with joint probability

   $$P(X = x, Y = y) = \frac{n!}{x!y!(n-x-y)!} p^x \theta^y (1-p-\theta)^{n-x-y}$$

   for x and y non-negative integers with $x + y \leq n$. You may assume that the joint probability generating function is $G_{X,Y}(s,t) = (ps + \theta t + (1-p-\theta))^n$.

   Obtain the distribution, and state the mean and variance, for each of X and Y. Obtain the coefficient of correlation $\rho(X,Y)$. Under what condition does $\rho(X,Y) = -1$? When this condition holds state the relationship between X and Y.

Turn Over . . .

3. (a) Two customers start being served at the same time. If X is the shorter of the service times and Y is the longer service time then $f_{X,Y}(x,y) = 2xye^{-(x+y)}$ for $0 < x < y < \infty$.

   Let $U = Y - X$ and $V = X + Y$. Find the joint p.d.f. for U and V and hence find the marginal p.d.f. for V.

   Find the conditional p.d.f. for $U|V = v$.

   (b) Suppose that $X|Y = y$ has uniform distribution with $f_{X|Y}(x|y) = \frac{1}{y}$ for $0 < x < y$ and $f_{X|Y}(x|y) = 0$ elsewhere. State $E[X|Y = y]$. Use the result that $E[XY] = E[YE[X|Y]]$ to show that $Cov(X,Y) = \frac{1}{2}Var(Y)$.

   If Y has p.d.f. $f_Y(y) = ye^{-y}$ for $y > 0$, obtain $Cov(X,Y)$.

4. If n is a positive integer and $Y \sim Gamma(\theta, n)$ then you may assume that the p.d.f. is $f_Y(y) = \frac{\theta^n y^{n-1} e^{-\theta y}}{(n-1)!}$ for $y > 0$ and the m.g.f. is $M_Y(t) = \left(1 - \frac{t}{\theta}\right)^{-n}$ for $t < \theta$.

   (a) Let $Z_1, ..., Z_k$ be independent with $Z_j \sim Gamma(\theta, n_j)$ (with n_j a positive integer) for $j = 1, ..., k$. Obtain the m.g.f. for $W = \sum_{j=1}^{k} Z_j$ and hence state the distribution of W.

   (b) Let $X \sim Gamma(\theta, m)$ be independent of $Y \sim Gamma(\theta, n)$. If $U = \frac{X}{Y}$ and $V = Y$, find the joint p.d.f. for U and V. Hence find the marginal p.d.f. for U.

   If $n > 1$ show that $E[U] = \frac{m}{n-1}$.