Queen Mary, University of London

B.Sc. Examination by course units

MAS228 Probability II

Duration: 2 hours

Date and time: 2nd May 2007, 10-12

The paper has two sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Electronic calculators may be used. The make and model should be specified on the script. The calculator must not be preprogrammed (other than by the manufacturer) prior to the examination.

You should not start reading this paper until instructed to do so by the invigilator.

You must not remove the question paper from the examination room.

Section A: You should attempt all questions. Marks awarded are shown next to the question. This part of the examination carries 60 % of the marks.

1. Let X be a non-negative integer-valued random variable which has probability generating function $G_X(t) = \frac{1}{3}(2+t^2)e^{t-1}$

   [4] (a) Find $E(X)$ and $Var(X)$.

   [3] (b) Find $P(X=0)$, $P(X=1)$ and $P(X=2)$.

[5] 2. A lift moves at random up and down between floors in a building with 10 floors, including the ground floor, numbered $0,1,\ldots,9$. Except when the lift reaches either the top or ground floor, each time it reaches a floor it is equally likely to go either up or down one floor.

   Gonzo enters the lift on the third floor. Find the probability that he reaches the top floor (the ninth floor) before he reaches the ground floor (floor zero).

[7] 3. State the law of total probability for expectations.

   Kermit plays a series of independent games. At the start of each game he pays £1 then rolls a fair 6-sided die. If he obtains a 6 he receives £k; otherwise he receives nothing. The games continue until he throws a 1, when the series of games stop. Find the expected amount he wins and hence state the value of k for which the game is fair (the expected amount he wins is zero).

4. Each male in a certain society has at most two sons and is equally likely to have 0, 1 or 2 sons, independently of all other males. Consider the male line of descent of a specific male.

   [4] (a) Find the probability that this male has no grandsons through the male line of descent.

   [4] (b) Find the probability that his male line of descent dies out eventually.

5. X and Y are independent chi-squared random variables with parameters n and m respectively and with moment generating functions $M_X(t) = (1-2t)^{-n/2}$ and $M_Y(t) = (1-2t)^{-m/2}$. Let $Z = X+Y$.

3 (a) Find $M_Z(t)$ and state the distribution of Z (including any parameters).

3 (b) Use the moment generating function to find $E[Z]$.

6. Random variables X and Y are jointly continuous with probability density function

$$f_{X,Y}(x,y) = \begin{cases} C(1+y^2) & \text{if } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

3 (a) Determine if X and Y are independent.

3 (b) Find C and the marginal density functions for X and Y.

3 (c) Find $E[XY]$.

7. Let X_1 and X_2 be random variables with $E[X_1] = 1$, $E[X_2] = 2$, $Var(X_1) = 1$, $Var(X_2) = 4$ and $Cov(X_1, X_2) = 1$. Define $Y_1 = X_1 + X_2$ and $Y_2 = 5X_1 - 2X_2$.

7 (a) Find $E[Y_1]$, $E[Y_2]$, $Var(Y_1)$, $Var(Y_2)$ and $Cov(Y_1, Y_2)$. Are Y_1 and Y_2 independent?

3 (b) If X_1 and X_2 have bivariate normal distribution, state the joint distribution of Y_1 and Y_2 (including any parameters).

5 8. (a) Let $X \sim Geometric\left(\frac{1}{2}\right)$. Use Markov's inequality to get an upper bound for $P(X \geq N)$ where N is a positive integer. Also find the exact probability.

3 (b) Let $X \sim Binomial(n, p)$. Use Chebyshev's inequality to get an upper bound for $P\left(\left|\frac{X}{n} - p\right| \geq \frac{p}{10}\right)$.

Section B: You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best TWO questions answered will be counted. This part of the examination carries 40% of the marks.

1. A gambler plays a sequence of independent games. At each game he has probability p of winning and probability $q = 1 - p$ of losing. He bets 1 unit each time, which is lost if he loses the game. If he wins the games he receives back his 1 unit bet plus an additional unit. He stops playing when he reaches N units or 0 units (in which case he goes broke).

   (a) Obtain the difference equations for L_k, the probability that he goes broke starting from k units, where k takes non-negative integer values with $k \leq N$. Solve the difference equations to obtain L_k for the case when $p \neq \frac{1}{2}$.

   (b) Let E_k be the expected duration of the game starting from k units. Obtain the difference equations for E_k where k takes non-negative integer values with $k \leq N$. Solve the difference equations to obtain E_k for the case when $p \neq \frac{1}{2}$.

2. (a) Let X and Y be discrete non-negative integer-valued random variables. Prove that $E[h(Y)] = E[E[h(Y)|X]]$. Use this result to derive expressions for the probability generating function $G_Y(t)$ and the moments $E[Y]$ and $Var(Y)$ in terms of conditional expectations.

   (b) The number of telephone calls N during the day at a call centre has Poisson distribution with parameter λ. Each call has a probability p of being a complaint, independent of the other calls. If Y counts the number of complaints in the day, state the conditional distribution of $Y|N = n$ and the probability generating function over the conditional distribution, $E[t^Y|N = n]$. Hence obtain the probability generating function $G_Y(t)$ and state the distribution of Y.

   (c) The number of customers N going to a store in a day has $E[N] = 500$ and $Var(N) = 25$. The amount of money X spent by a customer in a day has $E[X] = \mu$ and $Var(X) = \sigma^2$. This is independent of other customers and of N. Let X_j be the amount of money spent by customer j, so that $Y = \sum_{j=1}^{N} X_j$ is the total receipts for the store in the day. Find $E[Y]$ and $Var(Y)$.

3. Suppose that random variables X and Y are jointly continuous with probability density function

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-(x+y)} & \text{if } 0 < x < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

(a) i. Find the marginal density function for X. State the distribution of X.

ii. Find the conditional density function for $Y|X = x$ and show that $E[Y|X = x] = x + 1$.

(b) i. Find the joint probability density function for $U = Y - X$ and $V = X$.

ii. Show that U and V are independent and find their marginal density functions. State the distributions, means and variances for U and V.

(c) By using results in (a) or (b) or otherwise find $Cov(X,Y)$.

4. (a) If X and Y are independent continuous random variables, prove that $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$.

(b) $X \sim Gamma(\theta, \alpha)$ with p.d.f. $f_X(x) = \frac{\theta^\alpha x^{\alpha-1} e^{-\theta x}}{\Gamma(\alpha)}$ for $x > 0$ and $f_X(x) = 0$ elsewhere. Specify the range of t for which the moment generating function $M_X(t)$ is finite and show that for this range $M_X(t) = \left(1 - \frac{t}{\theta}\right)^{-\alpha}$.

(c) Z has double exponential distribution with probability density function $f_Z(z) = \frac{\theta}{2} e^{-\theta|z|}$ for $-\infty < z < \infty$. Find the moment generating function $M_Z(t)$.

(d) X and Y are independent with $X \sim Exp(\theta)$ and $Y \sim Exp(\theta)$.

i. Let $U = X + Y$. Find the moment generating function of U and hence state the distribution of U.

ii. Let $V = X - Y$. Find the moment generating function of V and hence state the distribution of V.

iii. By using the joint moment generating function or otherwise, show that U and V are not independent.

End of examination.