MTH5118 Probability II. Problem Sheet 9.

Please staple your coursework and post it in the Orange Box on the ground floor of the Maths building by **15:00** *on Wednesday 3rd December 2008.*

1. Let X and Y have joint m.g.f. $M_{X,Y}(s,t) = \frac{1}{(1-s)(1-s-t)}$. Find the univariate m.g.f. for each of X and Y and hence state the marginal distribution mean and variance for each of the random variables X and Y. Obtain the coefficient of correlation $\rho(X,Y)$.

Let $U = X - Y$ and $V = Y$. Find their joint m.g.f. and hence show that U and V are independent. Hence state the univariate m.g.f. and distribution for each of U and V.

2. $X \sim N(0,1)$ independent of $Y \sim N(0,1)$. Let $U = \mu + \sigma X$ and $V = \eta + \tau(\rho X + \sqrt{(1-\rho^2)}Y)$, where $\mu, \eta, \sigma, \tau, \rho$ are constants with $\sigma > 0$, $\tau > 0$ and $-1 < \rho < 1$.

(a) Find the means, variances and covariance for U and V and hence show that ρ is their coefficient of correlation.

(b) Find the joint m.g.f. for U, V. Obtain the m.g.f. and distribution of (i) U; (ii) V.

(c) Use 'transformation of variables' to show that (for all u, v) the joint p.d.f. of U, V is

$$f_{U,V}(u,v) = \frac{1}{2\pi\sigma\tau\sqrt{(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{u-\mu}{\sigma}\right)^2 - 2\rho\left(\frac{u-\mu}{\sigma}\right)\left(\frac{v-\eta}{\tau}\right) + \left(\frac{v-\eta}{\tau}\right)^2\right]}$$

(This is the bivariate normal p.d.f. so $M_{U,V}(s,t)$ is the m.g.f. for the bivariate normal.)

(d) Show that if $\rho = 0$ in the joint p.d.f. in (b) then U and V are independent. (Hence covariance zero implies independence for r.v.'s with bivariate normal distribution.)

(e) Use results in (b) and (c) to show that $V|U = u$ has normal distribution and state the conditional mean and variance.

3. X, Y and Z have joint p.d.f. $f_{X,Y,Z}(x,y,z) = 6\theta^3 e^{-\theta(x+y+z)}$ for $0 < x < y < z < \infty$. Find the marginal p.d.f. for Z. Hence find the conditional joint p.d.f. for $X, Y|Z = z$.

4. X, Y and Z have joint p.d.f. $f_{X,Y,Z}(x,y,z) = 6xyze^{-(x+y+z)}$ for $0 < x < y < z < \infty$. Find the joint p.d.f. for $U = X$, $V = Y - X$ and $W = Z - Y$. Are U, V and W independent?

5. $f_{X,Y|Z}(x,y|z) = \frac{2}{z^2}$ for $0 < x < y < z$ and $Z \sim Gamma(\theta, 3)$. Find (i) the joint p.d.f. of X, Y, Z; (ii) the joint p.d.f. of X, Y and their marginal distributions; (iii) the joint p.d.f. of X, Z and the conditional distribution of $Y|X = x, Z = z$.